The Cosmic Censorship Conjectures in General Relativity
or
Cosmic Censorship for the Massless Scalar Field with Spherical Symmetry

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Abstract

This essay describes Christodoulou's work on cosmic censorship for the massless scalar field model with spherical symmetry. First, the problem is introduced within the general framework for strongly tame models recently developed by Kommemi. The model is then described in greater detail and basic properties are proved. Next, Christodoulou's construction of solutions with naked singularities is summarised. Finally, the proof of weak cosmic censorship is given.

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1 Introduction

1.1 Cosmic Censorship at a Glance

Two of the most famous open problems in general relativity are the weak and strong cosmic censorship conjectures[1]. They are both statements about the generic behaviour of solutions to the Cauchy problem for the Einstein field equations. The weak conjecture roughly states that singularity formation happens behind the event horizon of a black hole. The strong conjecture roughly states that general relativity entirely predicts the future space-time

[1]Despite the nomenclature, the weak and strong conjectures are logically independent.

from initial data. It is important to emphasise that both conjectures may only be true in a generic sense, as certain "counterexamples" to both conjectures may be constructed.

1.1.1 The Cauchy Problem

The cosmic censorship conjectures are statements about *globally hyperbolic* solutions to the Cauchy problem in the theory of PDEs.

Definition A Lorentzian manifold is *globally hyperbolic* if every inextendible causal geodesic intersects a spacelike hypersurface Σ precisely once. In this case, we call Σ a Cauchy surface.

An important theorem for the theory of general relativity is that the Cauchy problem is well-posed for C^∞ initial data. More precisely,

Theorem 1.1 (Choquet-Bruhat and Geroch) *For a given reasonable matter model and reasonable C^∞ initial data specified on a spacelike surface Σ_0, there exists a unique maximal solution $(\mathcal{M},g_{ab},\phi^A)$ to the Einstein equations which is globally hyperbolic with Cauchy surface Σ_0 and which corresponds to initial data on $\Sigma_0 \subset \mathcal{M}$. This solution is maximal in the sense that $\mathcal{M}'$ for any other solution can be isometrically embedded into $\mathcal{M}$ so that the corresponding data ϕ'^A and ϕ^A coincide. We call $\mathcal{M}$ the maximal Cauchy development of Σ_0.*

The details of this theorem may be found in [Wal84] chapter 10.

Since space-time is itself dynamic and its general structure is not well understood, cosmic censorship in full generality is not well-posed. To formulate cosmic censorship, additional properties of solutions must be understood. For example, in the particular case of spherical symmetry, cosmic censorship is well-posed. This essay will focus entirely on solutions with spherical symmetry.

1.1.2 Spherical Symmetry

Imposing spherical symmetry simplifies cosmic censorship substantially–enough so that progress has already been made. Spherical symmetry greatly simplifies the Einstein equations as all angular derivatives vanish. Furthermore, $\mathcal{M}$ reduces to a $(1+1)$-dimensional quotient manifold $\mathcal{Q} := \mathcal{M}/SO(3)$, which can be conformally mapped to a bounded region of $\mathbb{R}^{1+1}$. Such a region (called a *Penrose diagram*) is not only useful visually, but also allows for a characterisation of the boundary using the topology induced by the embedding.

Familiarity with Penrose diagrams will be assumed during this essay, but a few important properties are noted here.

- ⋆ Penrose diagrams illustrate causal structure. Additionally, they give rise to a notion of *incoming* and *outgoing* null cones (often denoted resp. C^- and C^+), which are determined by their angle from horizontal (resp. 135^o and 45^o from right horizontal).
- ⋆ Every Penrose diagram contains a timelike curve Γ corresponding to the fixed points of the $SO(3)$ action on $\mathcal{M}$.[2] In light of the previous point, Γ appears on the left. All spheres on Γ have zero area.
- ⋆ Every Penrose diagram has an incoming null future boundary component $\mathcal{I}^+$ (called *future null infinity*), which emanates from the point i^0 (called *spacelike infinity*) corresponding to the boundary point of Σ_0, and terminates at the point $i^\square$ (called *timelike infinity*). By definition, the areas of spheres approaching $\mathcal{I}^+$ diverge. See figure 3.

1.1.3 The Conjectures

We are now prepared to discuss the cosmic censorship conjectures. Informally speaking, the weak cosmic censorship conjecture states that any space-time singularity is hidden inside a black hole. To formally define weak cosmic censorship, we first establish what it means for $\mathcal{I}^+$ to be *geodesically complete*.

Definition $\mathcal{I}^+$ is *geodesically complete* if the parallel transport of an inward-pointing null vector along an outgoing null geodesic approaching $\mathcal{I}^+$ yields a family of null geodesics with unbounded affine lengths. If this is the case, we denote $i^\square = i^+$ in the Penrose diagram. Otherwise, we denote $i^\square = i^{\text{naked}}$.

This definition is motivated by considering an observer so far away so as to be living *on* $\mathcal{I}^+$. This observer eventually sees everything in the causal past of $\mathcal{I}^+$ and it would be nice to say that this observer "lives forever" in some sense, which is the interpretation of geodesic completeness of $\mathcal{I}^+$. Weak cosmic censorship states that, generically, this is the case.

Conjecture 1.2 (Weak Cosmic Censorship Conjecture) *For generic regular initial data, $\mathcal{I}^+$ is geodesically complete.*

The strong cosmic censorship conjecture is concerned with the predictability of general relativity. Recall from Theorem 1.1 that there is a unique maximal Cauchy development $\mathcal{M}$ of an initial surface Σ_0. So on $\mathcal{M}$, we know that general relativity deterministically predicts the evolution of initial data. However, it may *a priori* be the case that $\mathcal{M}$ can be embedded into a larger

[2] This in general need not be true, but we shall only focus on solutions with initial hypersurface diffeomorphic to $\mathbb{R}^3$, for which this is true.

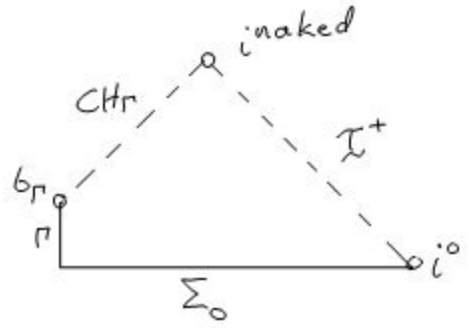


Figure 1: A penrose diagram exhibiting failure of weak cosmic censorship. Refer to §1.2.1 for a complete description of components.

manifold of which Σ_0 is not a Cauchy surface. The statement of strong cosmic censorship is that, generically, this does not actually happen.

Conjecture 1.3 (Strong Cosmic Censorship Conjecture) *For generic regular initial data, the maximal Cauchy development is future inextendible as a sufficiently smooth Lorentzian metric.*

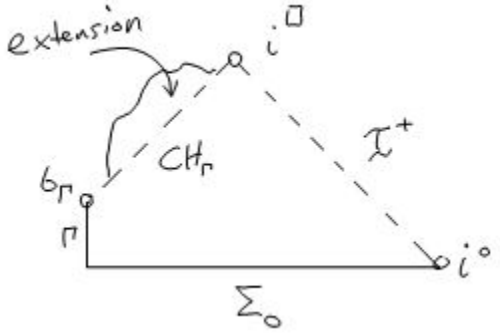


Figure 2: A penrose diagram exhibiting failure of strong cosmic censorship.

Future extensions must partly take place at a Cauchy horizon, which is a portion of the future boundary of the Penrose diagram (excluding $\mathcal{I}^+$) for which sphere areas do not continuously tend to zero.

Conjectures 1.2 and 1.3 have additional dependencies which must be addressed. First, there is some notion of genericity, which means the conjectures

must be considered for certain spaces of initial data. This issue will be addressed later. Second, the Einstein field equations are incomplete without an associated matter model. So conjectures 1.2 and 1.3 may be true or false for different types of matter.

1.2 The Matter Model

This essay will briefly explore a general class of matter models (the *strongly tame* models) and then focus on a particular one (the massless scalar field). The general class of matter models provides context in which results for the massless scalar field may be understood. These results are the primary focus of this essay.

1.2.1 Strongly Tame Models

In order to solve challenging problems in mathematics, it is helpful to focus on simpler ones which may provide some insight for the general solution. While this is especially true for the problem of cosmic censorship, it is also important when considering a particular model to consider it in the context of a larger class of models. Doing so provides a sense of what to expect and what in particular must be proved. This is the point of first considering the class of strongly tame matter models, which are described in detail in [Kom11].

The class of strongly tame models are those which satisfy the dominant energy condition and a *generalized extension principle*, which is defined as follows.

Definition The *generalised extension principle* states that for any p belonging to the future boundary $\overline{\mathcal{Q}} \setminus \mathcal{Q}$ such that $J^-(p) \cap (\overline{\mathcal{Q}} \setminus \mathcal{Q}) = \{p\}$ (which is to say that p is a *first singularity*), then any neighbourhood $N(p) \subset \mathbb{R}^{1+1}$ of p intersects $J^-(p) \setminus \{p\}$ at a region which either has infinite spacetime volume[3] or $\liminf r = 0$.

So,

Definition A matter model is *strongly tame* if it satisfies both the dominant energy condition and the generalized extension principle.

This particular choice of models is justified by the fact that it allows for general results to be proved while also including important models such as the

[3]It turns out this case can be excluded when additionally assuming the dominant energy condition.

Einstein-Maxwell-Klein-Gordon model, which describes a charged scalar field with mass.

Additionally, in [Kom11] and in this essay, only solutions will be considered which have asymptotically flat with one end initial data lacking anti-trapped surfaces (see §1.2.2).

The main result of [Kom11] is the characterisation of the future boundary of solutions to strongly tame models. This will be summarised here along with a brief description of the role of the trapped region in shaping the future boundary. A few properties listed below will be derived in §2.2.

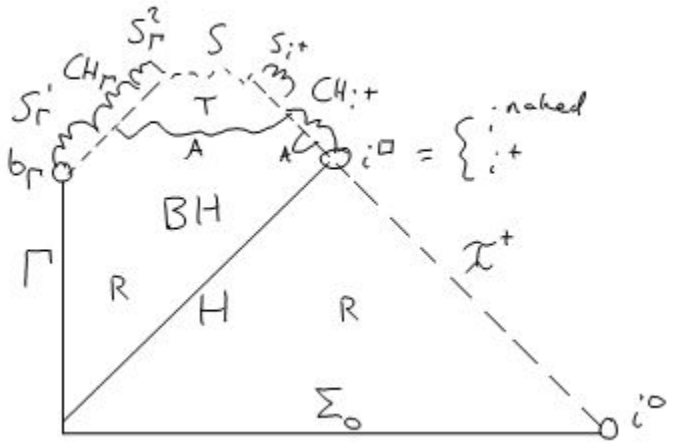


Figure 3: A comprehensive Penrose diagram for strongly tame matter models.

To describe the boundary components, we introduce the *area radius function* $r : \mathcal{Q} \to \mathbb{R}_{\geq 0}$.

$$4\pi r^2 := \text{Area},$$

where *Area* is the area of the corresponding sphere in $\mathcal{M}$. The future boundary of a strongly tame solution consists of some or all of the components shown in Figure 3. Those are

i_0 Spacelike infinity

$\mathcal{I}^+$ Future null infinity

$i^\square$ Timelike infinity, usually denoted i^+ (or i^{naked} in the case of a naked singularity.)

b_Γ The future limit point of the curve Γ corresponding to the centre of symmetry.

$S^1_\Gamma \cup CH_\Gamma \cup S^2_\Gamma$ The central component B_0 of the future boundary emanating from b_Γ. It is composed of three (possibly empty) outgoing null segments:

S^1_Γ and S^2_Γ are singular spheres where $r \to 0^+$ continuously, and CH_Γ is a Cauchy horizon where $r \to \tilde{r} > 0$ continuously.

S A (possibly empty) nowhere timelike curve where $r \to 0^+$ continuously.

$S_{i^+} \cup CH_{i^+}$ The future null component emanating from i^+. It is composed of two (possibly empty) outgoing null segments: S_{i^+} where $r \to 0^+$ continuously and CH_{i^+} where $r \to \tilde{r} > 0$.

The components are related to weak cosmic censorship according to the following proposition.

Proposition 1.4 *If either* $BH := \mathcal{M} \setminus J^-(\mathcal{I}^+)$ *is nonempty or* $\sup_{CH_\Gamma} r < \infty$, *then* $\mathcal{I}^+$ *is geodesically complete.*

1.2.2 The *RAT* Regions

The future boundary characterisation is related to the development of trapped surfaces. In a general (not necessarily spherically symmetric) spacetime, a closed 2-dimensional spacelike hypersurface admits two linearly independent future null normal vector fields. These induce two second fundamental forms χ and $\overline{\chi}$ on the surface. If the mean curvatures $\mathrm{Tr}\chi$ and $\mathrm{Tr}\overline{\chi}$ are negative everywhere, then the surface is called a *trapped surface*.

Trapped surfaces are of particular interest in general relativity, because of the *Penrose incompleteness theorem*:

Theorem 1.5 (Penrose) *Let* $(\mathcal{M}, g)$ *be globally hyperbolic with non-compact Cauchy surface and satisfying the null energy condition. If* $\mathcal{M}$ *contains a trapped surface, then* $(\mathcal{M}, g)$ *is future causally geodesically incomplete.*

The conclusion of the above theorem has obvious relation to cosmic censorship in the sense that the "place" at which geodesics may not be further extended in the maximal Cauchy development may either be a singularity or an event horizon admitting an extension. In the latter case, the non-generic formulation of strong cosmic censorship would be violated. Furthermore, the existence of a trapped surface implies the formation of a black hole, so by proposition 1.4 the formation of a trapped surface is sufficient to showing completeness of $\mathcal{I}^+$. This is how Christodoulou accomplishes his proof of weak cosmic censorship in [Chr99a].

In the case of spherical symmetry, we take the surfaces to be orbits of the $SO(3)$ action, and then it makes sense to call a point $p \in \mathcal{Q}$ trapped or marginally trapped. It turns out that p is trapped if $\partial_u r < 0$ and $\partial_v r < 0$ where ∂_u and ∂_v are ingoing and outgoing null derivatives. In the case of

spherical symmetry, a trapped surface implies a trapped sphere, so it suffices only to consider trapped spheres.

If $\partial_u r > 0$ and $\partial_v r > 0$ at a point p, then p corresponds to an *anti-trapped* sphere. But it will be explained (proposition 2.1) that $\partial_u r < 0$ always. We thus decompose $\mathcal{Q} = R \sqcup A \sqcup T$ where R, the *regular region* is the set of points at which $\partial_v r > 0$, while T, the *trapped region*, is the set of trapped spheres, and A, the *apparent horizon*, is the set of marginally trapped spheres. These regions, when referred to collectively, will be called the RAT regions.

In the case of strongly tame matter models, the quantity $\partial_v r$ has the property that if $\partial_v r = 0$ at a point p, then $\partial_v r \leq 0$ on the outgoing null ray emanating from p. Additionally, if $\partial_v r < 0$ at some point along this ray, then $\partial_v r < 0$ for the remaining (future) part of the ray. The reason for this will be explained in the following section (proposition 2.2), but it is mentioned at the moment to illustrate the general shape of the RAT regions. This illustration can be seen in Figure 4.

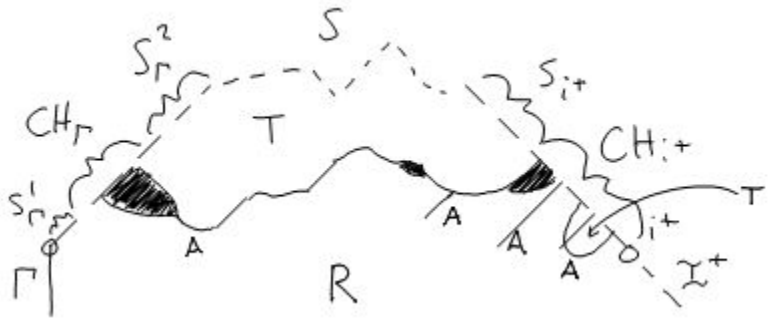


Figure 4: The possible behaviour of the RAT regions.

One particular consequence to the future boundary may already be seen. In a more general model, it is possible the central component might contain multiple disjoint Cauchy horizions $B_0 = S^1_\Gamma \cup CH^1_\Gamma \cup S^2_\Gamma \cup CH^2_\Gamma \cup S^3_\Gamma \cup ....$ However this is not the case for strongly tame matter models, because each outgoing null ray C^+ in $\mathcal{Q}$ may intersect the apparent horzion on at most one segment, and this intersection corresponds to a maximum of r along C^+.

In the case of Christodoulou's model, stronger monotonicity conditions are achieved for $\partial_u r$ and $\partial_v r$, which simplify the shape of the RAT regions and also the boundary (figure 5).

1.2.3 Christodoulou's Massless Scalar Field

Christodoulou's model [Chr87, Chr91, Chr93, Chr94, Chr99a] describes a massless scalar field. It belongs to the class of strongly tame models and is "simplified" enough that Christodoulou was able to prove both cosmic censorship conjectures. The term "simplified" appears in quotes, because the proof of cosmic censorship was anything but simple, and is considered to be a major accomplishment.

The five papers listed above each contribute to the understanding of the massless scalar field. In [Chr87] and [Chr91], Christodoulou investigates black hole formation. The latter paper provides a sufficient condition for initial data to lead to the formation of a trapped surface. This will be addressed soon (Theorem 2.7), and is critical for the proof of weak cosmic censorship.

The third paper [Chr93] shows that the Cauchy problem is well-posed in the bounded variation (BV) space (Theorem 2.3). This should be contrasted with Theorem 1.1, which holds only for smooth initial data. This result may be special to the case of spherical symmetry–it is related to the fact that the wave equation in $\mathbb{R}^{1+1}$ also has a well-defined Cauchy problem in BV space, although for higher dimensions this is not the case.

The fourth paper [Chr94] shows existence of particular solutions having boundaries with "bad behaviour," exhibiting a naked singularity, a Cauchy horizon with C^0 extension, and/or a singular future light cone. This itself does not disprove cosmic censorship provided that these traits are non-generic, but it shows that any proof of cosmic censorship must address this issue.

The complexity of Christodoulou's model is perhaps better appreciated after considering the fact that it is well-posed in a relatively general (BV) space and that solutions with naked singularities exist. Nevertheless, Christodoulou was able in [Chr99a] to prove weak cosmic censorship by showing that the set of solutions exhibiting naked singularities has codimension at least 2 in the BV space (Theorem 4.3), from which it follows a fortiori that naked singularity formation is non-generic.

*

The remainder of this essay summarises various parts of Christodoulou's work on the massless scalar field. Emphasis will be placed on the techniques employed to reach various conclusions and not particularly on the mathematical formalism required or proof. In §2 the relevant equations are introduced and used to justify various claims made in this introduction. The important Theorem 2.7 is stated at the end of this section. Next, §3 outlines the approach taken in [Chr94] to construct solutions with exotic boundaries. Finally, the method of proof of weak cosmic censorship is described in §4.

2 Mathematical Preliminaries

Here we introduce the problem at hand using a more mathematical footing. After introducing coordinates and the relevant equations, we derive a few results mentioned in the introduction which apply in general to all strongly tame matter models as well as a few results which hold particularly for the massless scalar field model. To prepare for more involved topics, we also discuss the Cauchy problem and an important theorem for gravitational collapse.

2.1 Coordinates

To begin analysis of Christodoulou's model, we first establish a coordinate system for $\mathcal{Q}$. Christodoulou often changes coordinates to adapt to various pictures, but this essay will primarily use double null coordinates, which are described in §2.1.1. Since Christodoulou frequently uses Bondi coordinates, these will be briefly discussed as well (§2.1.2).

2.1.1 Double Null Coordinates

Double null coordinates (u, v) are a natural choice of coordinates as they describe the causal structure of space-time. The coordinate u is chosen to be constant along outgoing light cones and the coordinate v is chosen to be constant along ingoing light cones. Such a choice determines u and v up to transformations $u \to f(u)$, $v \to g(v)$ for monotone f, g. To uniquely determine these coordinates, we fix[4] $v = 2r$ along an initial outgoing light cone C_0^+ and $u = v$ along Γ. This uniquely determines the double null coordinate system. Note that $(u, v) = (0, 0)$ at $\Gamma \cap C_0^+$. Note also that these coordinates determine a global chart on $\mathcal{Q}$ when the space-time $\mathcal{M}$ is globally hyperbolic.

Since the quotient manifold $\mathcal{Q}$ is locally conformal to $\mathbb{R}^{1+1}$, the metric on $\mathcal{Q}$ in double null coordinates takes the form

$$g_{ab}dx^a dx^b = -\Omega^2 dudv,$$

where the factor Ω^2 is called the *conformal factor*.

One particular reason to use double null coordinates is that they show the causal structure of the *singular boundary* B. We define the singular boundary to be $B := \overline{\mathcal{Q}} \setminus (\mathcal{Q} \cup i^0 \cup \mathcal{I}^+ \cup i^\square \cup CH_{i^+})$ where the closure $\overline{\mathcal{Q}}$ is in the

[4]This explicit choice of u and v is convenient mathematically, but has no physical meaning. In an abuse of terminology, it will be said that Penrose diagrams use double null coordinates, even though it is clear that they have been compactified using the aforementioned gauge freedom.

topology induced by a conformal embedding into $\mathbb{R}^{1+1}$. Additionally, we define $B_0 := \{(u_0, v) \in \overline{\mathcal{Q}} | u_0 = \sup_\Gamma u\}$ and call $B_0 \subset B$ the *central component* of the singular boundary. This is the outgoing null cone emanating from b_Γ. For some solutions, B_0 or even B may be empty (eg. Minkowski space).

2.1.2 Bondi Coordinates

Bondi coordinates (u, r) are used frequently by Christodoulou, especially when the focus is on a neighbourhood of b_Γ in the regular region. The coordinate u is defined to be constant along outgoing null geodesics and to vary according to proper time along Γ with $u = 0$ along B_0 (so $u < 0$ in $\mathcal{Q}$). The second coordinate r is the radial coordinate, which forms a chart in the regular region.

In Bondi coordinates, the metric takes the form

$$g_{ab}dx^a dx^b = -e^{2\tilde{\nu}}du^2 - 2e^{\tilde{\nu}+\tilde{\lambda}}dudr, \tag{1}$$

where $\tilde{\lambda}$ and $\tilde{\nu}$ are not to be confused with λ and ν, which will be defined later.

As previously mentioned, all diagrams in this essay will use double null coordinates. The drawback to using Bondi coordinates is that B_0 is collapsed to a single point, and various components of B_0 may be easily overlooked.

2.2 Initial Calculations

The model under consideration reduces to a quotient manifold $\mathcal{Q} \subset \mathbb{R}^{1+1}$, endowed with Lorentzian metric g_{ab} and two scalar fields ϕ and r corresponding to the massless scalar field and the area radius function. These quantities are solutions to the following non-linear system of PDEs

$$r\nabla_a\nabla_b r = \frac{1}{2}g_{ab}(1 - \partial^c r\partial_c r) - r^2 T_{ab} \tag{2a}$$

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial^c\phi\partial_c\phi$$

$$\nabla^a(r^2\partial_a\phi) = 0, \tag{2b}$$

where (2a) is Einstein's equation for the metric and (2b) is the wave equation for a massless scalar field. Recall that in double null coordinates, which form a global chart for $\mathcal{Q}$, the metric takes the form $-\Omega^2 dudv$. Then the system

(2) takes the following form

$$\partial_u(r\partial_v r) = -\frac{1}{4}\Omega^2 \tag{3a}$$

$$\partial_u(\Omega^{-2}\partial_u r) = -\Omega^{-2}r(\partial_u\phi)^2 \tag{3b}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -\Omega^{-2}r(\partial_v\phi)^2 \tag{3c}$$

$$r\partial_u\partial_v\phi + \partial_u r\partial_v\phi + \partial_v r + \partial_u\phi = 0. \tag{3d}$$

2.2.1 Results from Immediate Monotonicity

Immediately, we can see a few results due to the fact that the quantities $\Omega^{-2}\partial_u r$ and $\Omega^{-2}\partial_v r$ are nonincreasing in the incoming and outgoing directions respectively. First, we see that anti-trapped surfaces cannot form.

Proposition 2.1 *If there are no anti-trapped surfaces present in initial data, then there will be no anti-trapped surfaces in the entire future solution.*

Proof Note that from global hyperbolicitiy, every inextendible incoming null curve intersects the Cauchy hypersurface where $\partial_u r < 0$ by assumption. So if an anti-trapped surface were to form, the quantity $\partial_u r$ would have to change from negative to positive along an incoming null curve, but that would require $\Omega^{-2}\partial_u r$ to change from negative to positive, which is forbidden by equation (3b). ∎

A similar argument using equation (3c) provides constraints on the shape of the apparent horizon and trapped region.

Proposition 2.2 *Along any outgoing null curve, the regular region must entirely precede the apparent horizon (if non-empty) and the apparent horizon (if non-empty) must entirely precede the trapped region (if non-empty).*

Proof The proof is analogous to that of Proposition 2.1, using equation (3c) in place of equation (3b). ∎

The equations (3b) and (3c) are specific to Christodoulou's model, however the general case is not very different. In general, equations (3a)-(3c) take the form

$$\partial_u\partial_v r = -\frac{1}{4}\Omega^2 - \partial_u r\partial_v r + 4\pi r^2 T_{uv} \tag{4a}$$

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r\Omega^{-2}T_{uu} \tag{4b}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r\Omega^{-2}T_{vv}, \tag{4c}$$

from which it follows that certain monotonicity properties (attributed in general to Raychaudhuri) may be salvaged provided the null energy condition holds.

2.2.2 The Initial Value Problem

The system (3), along with regularity conditions on Γ, also allows for simplification of the initial value problem in Christodoulou's model. Suppose $\partial_v\phi$ and r are both known on an outgoing light cone C^+ emanating from Γ. Then it is possible to solve for Ω on C^+ by integrating equation (3c) in v. Having solved for Ω, it is then possible by equation (3a) to determine $\partial_u r$ along C^+. Finally, it is possible to solve for $\partial_u\phi$ using equation (3d). At this point, the entire solution $(r, \partial_v\phi, \partial_u\phi, \Omega)$ is known on C^+, which is sufficient given that ϕ need only be determined up to a constant.

Differentiating $\partial_u\phi$ along C^+, we obtain the quantity $\partial_u(\partial_v\phi)$, which, along with $\partial_u r$, determines the future evolution of $\partial_v\phi$ and r to later light cones. Thus, the entire solution on $\mathcal{Q}$ is determined by specifying r and $\partial_v\phi$ on an initial future light cone C_0^+ emanating from Γ. Furthermore, given the coordinate choice $v = 2r$ on C_0^+ from the introduction, then the entire solution is determined by $\partial_v\phi$. Christodoulou showed in [Chr93] that

Theorem 2.3 *Given initial data $\vartheta := (r\partial_v\phi)|_{u=0}$ belonging to the bounded variation class, then there is a corresponding unique maximal globally hyperbolic solution to the system of equations* (2)*.*

2.2.3 Additional Monotonicity

It is often convenient to eliminate the conformal factor Ω, using instead the Hawking mass m, defined to satisfy the following equation

$$1 - \frac{2m}{r} := g^{ab}\partial_a r\partial_b r = -4\Omega^{-2}\partial_u r\partial_v r.$$

Then the system (3) becomes

$$r\partial_u\partial_v r = \frac{\frac{2m}{r}}{1-\frac{2m}{r}}\partial_u r\partial_v r \tag{5a}$$

$$2\partial_u r\partial_u m = \left(1 - \frac{2m}{r}\right) r^2(\partial_u\phi)^2 \tag{5b}$$

$$2\partial_v r\partial_v m = \left(1 - \frac{2m}{r}\right) r^2(\partial_v\phi)^2 \tag{5c}$$

$$r\partial_u\partial_v\phi = -\partial_u r\partial_v\phi - \partial_v r\partial_u\phi \tag{5d}$$

In addition, we introduce the following characters[5]

$$\lambda := \partial_v r \qquad \nu := \partial_u r$$
$$\theta := r\partial_v \phi \qquad \zeta := r\partial_u \phi$$
$$\mu := \frac{2m}{r}.$$

Then,

Proposition 2.4 *The following inequalities hold in the regions specified.*

Quantity	R	A	T
ν $(= \partial_u r)$	< 0 *everywhere*		
λ $(= \partial_v r)$	> 0	$= 0$	< 0
$1 - \frac{2m}{r}$	> 0	$= 0$	< 0
$\partial_u m$	≤ 0	$= 0$	≥ 0
$\partial_v m$	≥ 0 *everywhere*		
μ $(= \frac{2m}{r})$	≥ 0 *everywhere*		
$\partial_u \lambda$	≤ 0	< 0	< 0
$\partial_v \nu$	≤ 0	< 0	< 0

Proof The inequality $\nu < 0$ was already established (proposition 2.1). The inequalities for λ hold by definition of the RAT regions. The inequalities for $1 - \frac{2m}{r} = -4\Omega^{-2}\nu\lambda$ then immediately follow. For monotonicity properties on m, we turn to equations (5b) and (5c), which take the form

$$2\partial_u m = \frac{1-\mu}{\nu}\zeta^2, \qquad \text{and} \qquad 2\partial_v m = \frac{1-\mu}{\lambda}\theta^2.$$

Using Ω in place of m, the latter equation becomes $2\partial_v m = -4\Omega^{-2}\nu\theta^2$, and so $\partial_v m \geq 0$ everywhere. Then it follows by integration along outgoing null curves from Γ (since $m = 0$ on Γ by regularity assumptions) that $m \geq 0$ and hence $\mu \geq 0$. Lastly, from equation (5a) we have

$$\partial_u \lambda = \partial_v \nu = \frac{\mu}{1-\mu}\frac{\lambda\nu}{r} = -\frac{\mu\Omega^2}{4r}.$$

Note that $\mu\Omega^2 = 0$ only when $0 = \mu < 1$, which happens only in R. This shows the monotonicity properties of λ and ν. ∎

[5]Since calculations are not a significant part of this essay, I was unsure whether to define these quantities at all. I ultimately decided to use them, because Christodoulou seemed to have adopted them in his later papers and they are also used by Kommemi.

Proposition 2.4 is useful for establishing additional properties of solutions to Christodoulou's model. For example, a result simplifying A and a result simplifying B_0 are proved below.

Proposition 2.5 *The apparent horizon is an achronal, connected set and has no incoming null segments.*

Proof There cannot be incoming null segments, because this would require $\partial_u r = 2\partial_u m$ as $r = 2m$ on A, but $\partial_u r < 0$ and $\partial_u m = 0$ on A. So every incoming null curve which intersects A must intersect A at a single point p and the incoming null curve emanating from p (and excluding p) must belong to the trapped region as $\partial_u \lambda < 0$ on A. Furthermore, since we know the outgoing null curve emanating from p must be in $A \cup T$, then we see that $J^+(p) \in A \cup T$. Thus, A is achronal.

Suppose A is not connected. Then there is an incoming light cone $C^- \subset R$ which divides A into two sets. This light cone must terminate somewhere on the future boundary. Consider the outgoing light cone C^+ terminating on the same point of the boundary. (If C^- terminates on an outgoing null segment, then we may without loss of generality assume it terminates at the first singularity, the same point where C^+ terminates.) Clearly, $C^+ \cap T = \emptyset$, so $C^+ \cap C^- \in \overline{\mathcal{Q}}$ must have positive radius. But $C^+ \cap C^-$ is a first singularity, which means $\liminf r = 0$ by the generalised extension principle. This is a contradiction. ∎

The apparent horizon now looks like A in figure 5.

Proposition 2.6 *Consider a null segment $I = \{(u_1, v) | v \in [v_0, v_1]\}$ on the future boundary which borders the trapped region and contains no limit points of A. The function $r(u_1, v) := \lim_{u \to u_1^-} r(u, v)$ exists and must be strictly decreasing in v.*

Proof It is clear that the function $r(u_1, v)$ is well defined, since $\partial_u r < 0$ and $r \geq 0$. It is also clear that $r(u_1, v)$ must be non-increasing on the segment I, because it borders the trapped region. It must only be shown that $r(u_1, v)$ cannot be constant anywhere on this segment. This shall be established using a contradiction argument.

Suppose $r(u_1, v)$ is constant on some $I_0 \subset I$. Without loss of generality, let $I_0 = I$. Then $r(u_1, v_0) = r(u_1, v_1)$. By assumption, there is a $u_0 < u_1$ so

that $[u_0, u_1) \times [v_0, v_1] \subset T$. Then we calculate

$$\begin{aligned} r(u_0, v_1) - r(u_0, v_0) &= \left[r(u_1, v_1) - \int_{u_0}^{u_1} \nu(\tilde{u}, v_1) d\tilde{u} \right] - \left[r(u_1, v_0) - \int_{u_0}^{u_1} \nu(\tilde{u}, v_0) d\tilde{u} \right] \\ &= \int_{u_0}^{u_1} (\nu(\tilde{u}, v_0) - \nu(\tilde{u}, v_1)) d\tilde{u} \\ &\geq 0 \end{aligned}$$

But also, $r(u_0, v_1) - r(u_0, v_0) = \int_{v_0}^{v_1} \lambda(u_0, \tilde{v}) d\tilde{v} < 0$ since $\{u_0\} \times [v_0, v_1] \subset T$. Thus, we have a contradiction. ∎

In particular, Proposition 2.6 shows that the component S^2_Γ is empty for Christodoulou's model, because it has $r = 0$ and must border the trapped region. Thus, the central boundary component looks like B_0 in figure 5.

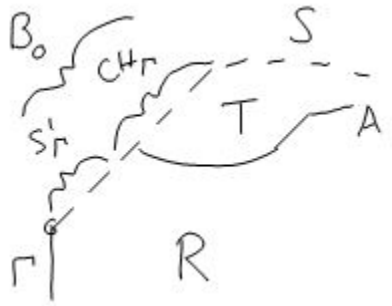


Figure 5: The apparent horizon and central boundary component are simplified by propositions 2.5 and 2.6.

2.3 A Theorem of Gravitational Collapse

So far, we have discussed consequences of a non-empty apparent horizon. To conclude this section, we shall briefly discuss the primary result of [Chr91], which provides sufficient conditions for initial data on C_0^+ to guarantee the existence of a non-empty apparent horizon.

Theorem 2.7 *Let C_0^+ be an outgoing future light cone with vertex on Γ and consider the annular region in C_0^+ bounded by two spheres $S_{1,0}$ and $S_{2,0}$ with $S_{2,0}$ in the exterior of $S_{1,0}$. Let C_1^- and C_2^- be the incoming future light cones passing through $S_{1,0}$ and $S_{2,0}$. Define the dimensionless size δ_0 and mass content η_0 by*

$$\delta_0 := \frac{r_{2,0}}{r_{1,0}} - 1, \text{ and } \eta_0 := \frac{2(m_{2,0} - m_{1,0})}{r_{2,0}}.$$

Then there are positive constants $c_0 \leq 1/e$ and $c_1 \geq 1$ such that if

$$\delta_0 \leq c_0 \text{ and } \eta_0 > c_1 \delta_0 \log\left(\frac{1}{\delta_0}\right),$$

then there exists an outgoing light cone C_*^+ with vertex on Γ that intersects C_1^- at a sphere with positive radius and intersects C_2^- at a marginally trapped sphere.

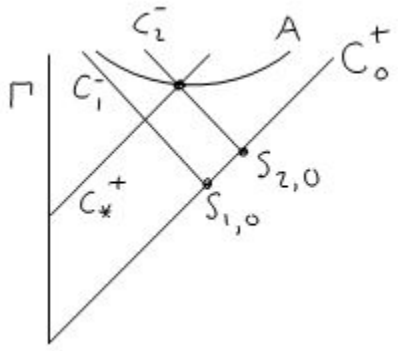


Figure 6: An illustration for Theorem 2.7.

Since existence of a marginally trapped surface is sufficient to show completeness of $\mathcal{I}^+$, this result will be very important in §4.

3 Exotic Solutions

In [Chr94], Christodoulou constructs solutions to the spherically symmetric massless scalar field model. Some of these solutions are of particular interest to the study of cosmic censorship, because they exhibit singular behaviour on the boundary including naked singularities and Cauchy horizons admitting extensions.

This section describes Christodoulou's construction, which involves reducing a certain family of solutions (the self-similar family) to a first order ODE system (§3.1), analyzing the critical points of the ODE system (§3.2) which characterise future boundary behaviour (§3.3), and finally, modifying initial data outside a suitably large region (§3.4) to impose asymptotic flatness and influence behaviour on the central component B_0.

3.1 Self-Similar Solutions and the Scaling coordinate

We first note that the space of solutions to the system (2) is preserved by the following 2-parameter group of transformations

$$g_{ab} \to a^2 g_{ab}, \qquad r \to ar, \qquad \phi \to \phi - b \tag{6}$$

where $a > 0$ and $b \in \mathbb{R}$. In order to construct his solutions, Christodoulou examines the 1-parameter subgroup fixed by constant $k \neq 0$ for which $b = -k \log a$. More specifically, he focuses on the class of solutions for which there is a corresponding 1-parameter group of diffeomorphisms f_a acting on $\mathcal{Q}$ such that

$$a^2 g_{ab} = f_a^* g_{ab}, \qquad ar = f_a^* r, \qquad \phi - k \log a = f_a^* \phi \tag{7}$$

which he defines to be the *self-similar* solutions.

Let the family f_a be generated by the vector field $\vec{S}$. Christodoulou shows that in Bondi coordinates,

$$\vec{S} = u\partial_u + r\partial_r \tag{8}$$

and furthermore that the conditions (7) imply

$$\vec{S}\tilde{\nu} = \vec{S}\tilde{\lambda} = 0 \text{ and } \vec{S}\phi = -k, \tag{9}$$

where $\tilde{\lambda}$ and $\tilde{\nu}$ determine the metric in Bondi coordinates (equation (1)). Note that the diffeomorphisms f_a have a fixed point at $(u, r) = (0, 0)$. We call this the scaling origin. From (7), we see that ϕ must diverge at this fixed point, so it corresponds to a singularity, and more particularly, to b_Γ.

Define the quantities[6]

$$s := \log(-r/u), \qquad \beta := 1 - \frac{1}{2} e^{\tilde{\nu} - \tilde{\lambda} - s}, \qquad \theta := r\partial_r \phi. \tag{10}$$

From (8) and (9), it is straightforward to verify that the quantities (s, β) are constant, and additionally since $[\vec{S}, \partial_r] = \partial_r$, that θ grows exponentially along integral curves of $\vec{S}$. That is

$$\vec{S}s = \vec{S}\beta = 0, \text{ and } \vec{S}\theta = \theta. \tag{11}$$

We shall treat s as a complementary coordinate, the *scaling coordinate*, which is constant on integral curves of $\vec{S}$ and for which $s = -\infty$ on Γ and $s = \infty$ on the central boundary component. See figure 7.

[6]In this section and in this section only, the quantity θ is redefined to be $\theta := r\partial_r \phi$. The main reason is to maintain compatibility with [Chr94], in which θ is used extensively. Furthermore, since r replaces v in Bondi coordinates, both θs in this paper represent morally the same thing.

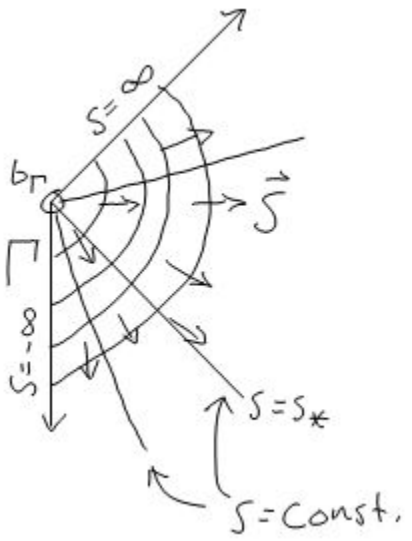


Figure 7: The scaling coordinate in double null coordinates

Christodoulou shows that $\tilde{\lambda}$ and $\tilde{\nu}$ can be expressed in terms of k, β, and θ, which means an entire solution can be described by the pair (θ, β). *This fact, along with* (11)*, allows us to express the self-similar solutions as solutions to a pair of ODEs:* Since the scaling coordinate s sweeps $\mathcal{Q}$, then by solving $(\beta(s), \theta(s))$ on a single curve transverse to $\vec{S}$, we specify the entire solution via (11). Christodoulou explicitly derives the system of ODEs from (2) to be the following:

$$\frac{d\beta}{ds} = 1 - k^2 - [(\theta + k)^2 + 1 - k^2]\beta, \tag{12a}$$

$$\frac{d\theta}{ds} = \frac{k}{\beta}(k\theta - 1) + [(\theta + k)^2 - (1 + k^2)]\theta. \tag{12b}$$

It is from this starting point that Christodoulou constructs his zoo of exotic solutions.

3.2 Exploring the Self-Similar Solutions

In §3.1, we concluded that a self-similar solution to (2) corresponds to a curve with parameter s on the (θ, β) plane satisfying the ODE system (12). Moreover, the parameter s corresponds to the scaling coordinate on $\mathcal{Q}$.

The purpose of §3.2 is to examine the properties of these curves. Regularity conditions on the metric at $\Gamma = \{s = -\infty\}$ require $\beta \to -\infty$ as $s \to -\infty$. It

turns out that for solutions which interest us, $\beta \to 0$ as $s \to s_*$ for a particular $s_* < \infty$ corresponding to the ingoing null ray terminating at b_Γ. Furthermore, $\beta > 0$ for $s > s_*$. For this reason, we call solutions corresponding to the $s < s_*$ portion of the curve in the lower half (ie. $\beta < 0$) plane *interior solutions*, and we call solutions corresponding to the $s > s_*$ portion of the curve in the upper half plane *exterior solutions*. We shall explore these solutions separately.

3.2.1 Interior Solutions

Interior solutions are discussed in detail in Section 1 of [Chr94]. The main ideas may be summarised as follows:

1. Either $k^2 < 1$, in which case (β, θ) is defined on $s \in (-\infty, s_*)$ and $\lim_{s \to s_*}(\beta, \theta) = (0, 1/k)$ (called the *singular point*) or $k^2 \geq 1$ in which case (β, θ) is defined on $(-\infty, \infty)$.

2. It is shown in the last section of [Chr94] that s_* corresponds to the boundary of $J^-(b_\Gamma)$.

3. In light of (1) and (2), for $k^2 \geq 1$, the solutions do not contain the scaling origin b_Γ. So this case is uninteresting for singularity formation. Therefore, we restrict our attention to the case $k < 1$.

As remarked earlier, regularity conditions on the solution at Γ provide initial conditions $(\theta, \beta) \to (0, -\infty)$ as $s \to -\infty$. To more clearly understand these initial conditions and for purposes which will become clear later on, it is helpful to replace β with its multiplicative inverse

$$\alpha := 1/\beta. \tag{13}$$

Then the ODE system (12) becomes

$$\frac{d\alpha}{ds} = \alpha[(\theta + k)^2 + (1 - k^2)(1 - \alpha)], \tag{14a}$$

$$\frac{d\theta}{ds} = k\alpha(k\theta - 1) + \theta[(\theta + k)^2 - (1 + k^2)]. \tag{14b}$$

And the initial conditions become $\lim_{s \to -\infty}(\theta, \alpha) = (0, 0)$. This point $(0, 0)$ is an unstable critical point. Note that since $k^2 < 1$, for $\beta < 0$ (equivalently, $\alpha < 0$) we have $\frac{d\alpha}{ds} < 0$ and any integral curve emanating from the origin will tend toward $\alpha \to -\infty$. This is just the statement that the interior solution ends at the critical point.

To conclude, *the interior solutions are characterised by a 1-parameter family of curves beginning at* $(\theta, \alpha) = (0, 0)$ *and asymptotically approaching* $\theta = 1/k$ *in the negative-*α *direction.*

3.2.2 Exterior Solutions

The exterior solutions correspond to curves originating at the singular point and lying entirely in the upper half (θ, α) plane shown in Figure 8. In describing this plane, we may assume $k > 0$, since the system (14) is invariant under the transformation $k \to -k, \theta \to -\theta$. The plane may be divided into regions based on the signs of $\frac{d\alpha}{ds}$ and $\frac{d\theta}{ds}$.

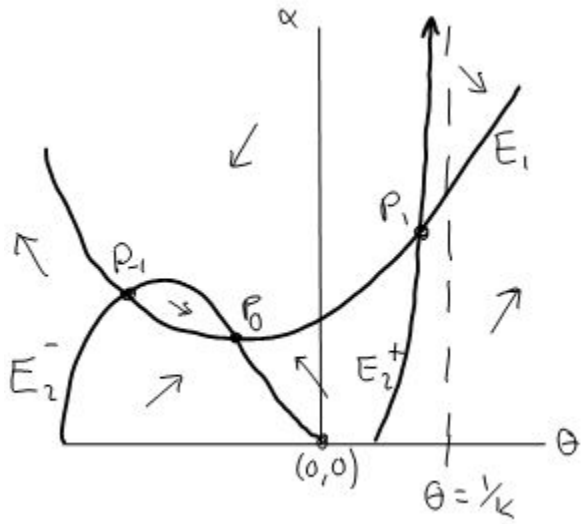


Figure 8: The upper half plane ($\alpha \geq 0$)

The derivative $\frac{d\alpha}{ds}$ changes sign along a single curve E_1 (excluding the line $\alpha = 0$), below which it is positive and above which it is negative. The derivative $\frac{d\theta}{ds}$ changes sign along two curves E_2^+ and E_2^-. It is positive to the right of E_2^+ and negative to the left of E_2^+ except for the region bounded by E_2^- and the line $\alpha = 0$ in which it is again positive. E_2^+ has an asymptote at $\theta = 1/k$. For more detail on the properties of this upper-half plane, see [Chr94] section 3.

There are three critical points denoted P_{-1}, P_0, and P_1 from left to right at the intersections of the curve E_1 with the curves $E_2^{\pm}$. The point P_0 is an attracting critical point and the points $P_{\pm 1}$ are hyperbolic. All three of these points lie to the left of the asymptote line $\theta = 1/k$ for positive values of k.

Curves in the upper half plane originating from the singular point exhibit asymptotic behaviour, approaching the line $\theta = 1/k$ with $\alpha \to \infty$ as $s \searrow s_*$. As s increases, these curves either terminate at a critical point or they diverge to $(-\infty, \infty)$ or (∞, ∞). To be more specific, we must distinguish between the cases $k^2 < 1/3$ and $k^2 \geq 1/3$.

For the case $k^2 < 1/3$ (see Figure 9, left), there is a single curve terminating at P_{-1} and a single curve terminating at P_1. This is to be expected, because $P_{\pm 1}$ are hyperbolic points. Between these two curves is a class of curves terminating at P_0. The remaining curves either diverge to $(-\infty, \infty)$ or to (∞, ∞).

For the case $k^2 \geq 1/3$ (see Figure 9, right), there is a single curve terminating at P_1, but the points P_{-1} and P_0 are "blocked" by the unstable manifold (shown as a darker curve) emanating from P_1. The remaining curves either diverge to $(-\infty, \infty)$ or to (∞, ∞).

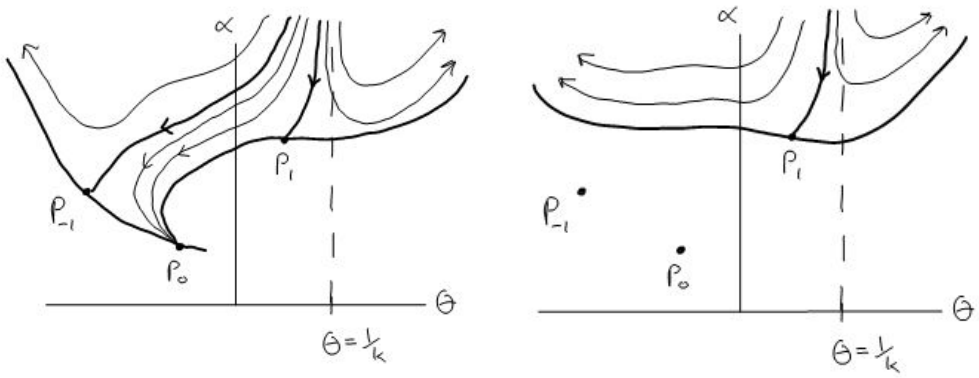


Figure 9: Exterior solutions for $k^2 < 1/3$ (left) and $k^2 > 1/3$ (right). The darker lines correspond to stable and unstable submanifolds, which divide the classes of solutions.

In summary, *the exterior solutions are characterised by their end behaviour. They correspond to curves which originate as an asymptote at $\theta = 1/k$, with $\alpha \to \infty$ as $s \searrow s_*$ and which end in one of the following ways as s increases:*

1. *A family of curves diverging $(\theta, \alpha) \to (-\infty, \infty)$*
2. *A single curve terminating at P_{-1} for the case $k^2 < 1/3$*
3. *A family of curves terminating at P_0 for the case $k^2 < 1/3$*
4. *A single curve terminating at P_1*
5. *A family of curves diverging $(\theta, \alpha) \to (\infty, \infty)$.*

As a remark, there is no one-to-one correspondence between interior solutions and exterior solutions as described above. Therefore, entire solutions

belong to either 1-parameter or 2-parameter families with one parameter corresponding to the interior solution and the other parameter (excluding the case of curves terminating at $P_{\pm 1}$) corresponding to the exterior solution.

3.3 Future Boundary Behaviour

In §3.2, we classified curves in $(\theta, \alpha = 1/\beta)$ space relevant to self-similar solutions on $\mathcal{Q}$. In particular, these curves may be characterised by their end behaviour as $s \to \infty$. This section describes properties of the central boundary component of these solutions in $\overline{\mathcal{Q}}$. Most of the statements in §3.3 are explained in more detail at the end of [Chr94].

3.3.1 Diverging Curves

Christodoulou shows that curves which diverge to $(-\infty, \infty)$ and (∞, ∞) do so in some finite interval $s \in (s_*, s_\infty)$, $s_\infty < \infty$. In $\mathcal{Q}$, the curve $\{s = s_\infty\}$ corresponds to an apparent horizon emanating from b_Γ. In this case, singularity formation is uninteresting as it will be contained inside the black hole.

3.3.2 Curves Terminating at $P_{\pm 1}$

As concluded in §3.2, there is a 1-parameter family of solutions which terminate at P_1. In the additional case $k^2 < 1/3$, there is also a 1-parameter family of solutions terminating at P_{-1}. The entire future (central) boundary component is a future null singular boundary. The radius function r vanishes on this boundary, so it is not a naked singularity. Each point of this boundary in $\overline{\mathcal{Q}}$ corresponds to a single point in the causal boundary of $\mathcal{M}$. (This is the very last remark in [Chr94].) The wave function ϕ diverges as $\log(r)$ on the boundary. Expressing the metric in null coordinates $g_{ab}dx^a dx^b = -\Omega^2 du dv$, the conformal factor Ω^2 at the boundary vanishes in the P_1 case and diverges in the P_{-1} case.

3.3.3 Curves Terminating at P_0

As concluded in §3.2, in the case $k^2 < 1/3$, there is a 2-parameter family of solutions which terminate at P_0. The Gauss curvature, which is continuous on $\mathcal{Q}$, diverges as r^{-2} at b_Γ. The point b_Γ is an isolated singular point and the future boundary component emanating from b_Γ is a null cone. The radial function r diverges along this future null cone. Therefore, *the singular point b_Γ would be a naked singularity if the space-time were asymptotically flat.*

There is yet another peculiar aspect of this family of solutions. The metric on $\{s = \infty\}$ coincides with the metric at $\{s = s_*\}$. Also, the null derivatives θ and ζ of ϕ on $\{s = \infty\}$ correspond respectively to the null derivatives ζ and θ of ϕ on $\{s = s_*\}$. Therefore, the past boundary of the time-reversed interior solution can be attached to the future boundary of the exterior solution in a C^0 manner (figure 10). In other words, *the future boundary admits a C^0 extension.*

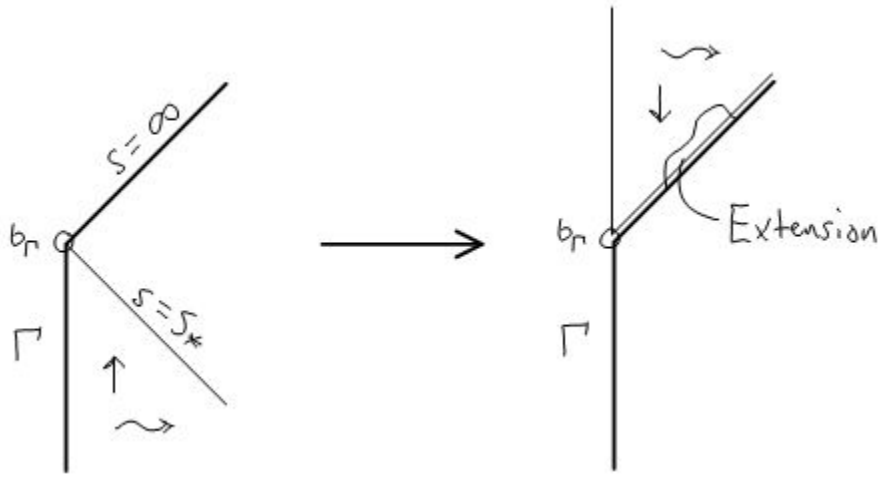


Figure 10: A construction for a C^0 extension at the future boundary of a globally hyperbolic solution.

3.4 Christodoulou's Exotic Zoo

From the previous section, we understand the kinds of singularities which may be constructed from self-similar solutions. However, these solutions are not asymptotically flat, which can be seen from the fact that the Hawking mass diverges as $r \to \infty$ with fixed $u = u_0$ for any $u_0 < 0$. Since we are interested in solutions with asymptotically flat initial data, there is still some work to be done. In this final §3.4, we discuss a recipe for constructing various types of asymptotically flat solutions with similar boundary behaviour.

3.4.1 Modifying Initial Data

The procedure used in [Chr94] is most easily described using double null coordinates and is depicted in Figure 11. Pick a point (u_0, v_0) in the exterior. Let u^* correspond to $\{s = \infty\}$. Specify initial data on the set $[u_0, u^*) \times \{v_0\}$ so as

to correspond to the self similar solution and then specify initial data on the set $\{u_0\} \times [v_0, \infty)$ so as to manipulate the boundary behaviour at $\{u^*\} \times (v_0, \infty)$ as desired.

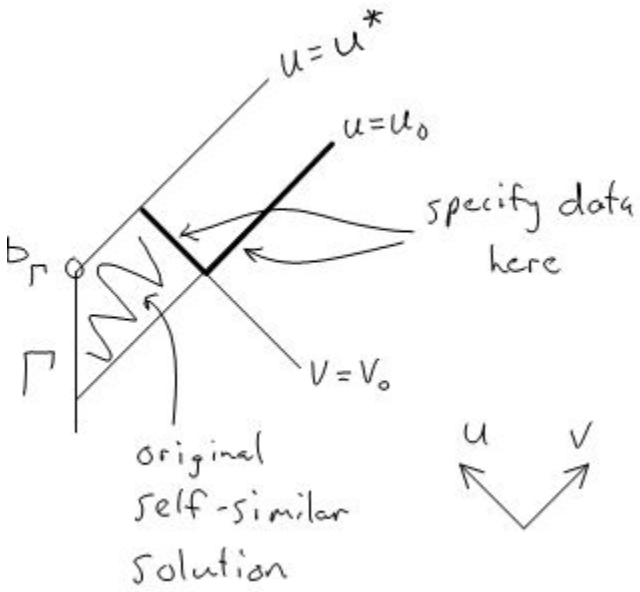


Figure 11: Modifying the self-similar solution to obtain an asymptotically flat solution.

This yields a solution to the Cauchy problem with initial surface $\{u = u_0\}$ *and having boundary behaviour corresponding to the self-similar solution for* $v < v_0$*, but behaving as desired as* $v \to \infty$*.*

3.4.2 The Examples

According to the prescription for flattening self-similar solutions, it is possible to construct five different types of boundary from solutions terminating at P_0 or $P_{\pm 1}$. For reference, see figure 12.

For a self-similar solution terminating at P_0, the initial boundary component takes the form CH_Γ. Excluding the case of trapped surface formation, the radial coordinate must be nondecreasing along B_0. According to proposition 1.4, if it diverges, then b_Γ is a true naked singularity (case A in Figure 12), but if it is bounded above, then $\mathcal{I}^+$ is complete (case B in Figure 12). Note that both solutions admit local C^0 extensions on the boundary part which coincide with the self-similar solution. In fact, Christodoulou modifies the self similar

solutions in a particular way so that there is a C^0 extension along the entire central boundary component.

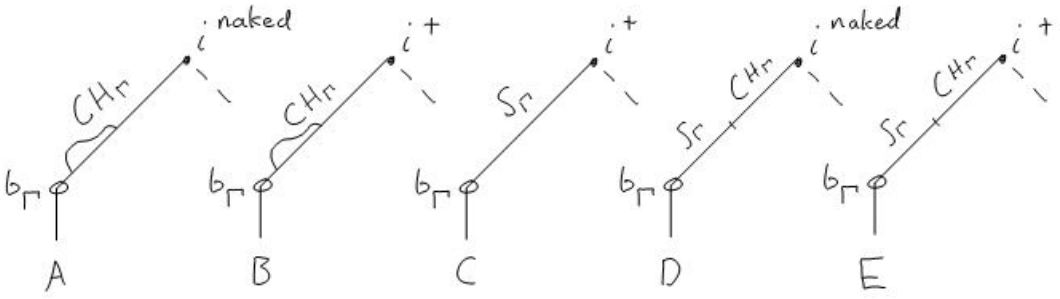


Figure 12: The five types of future boundary central components which may be constructed.

For a self-similar solution terminating at $P_{\pm 1}$, the initial boundary component takes the form S_Γ. Along the modified portion of B_0, the radial coordinate may be entirely zero (case C in Figure 12), may increase indefinitely (case D in Figure 12), or may increase with an upper bound (case E in Figure 12). Thus, cases D and E also have CH_Γ components and case D exhibits a naked singularity.

4 Instability of Naked Singularities

The examples in §3 were bad news for weak cosmic censorship, because some solutions feature naked singularities. However, it turns out that these solutions are unstable, and more importantly, the set of all such solutions is non-generic in BV. This is precisely the statement of weak cosmic censorship, and so the proof of this statement is the final topic of this essay. The details pertaining to this section may be found in [Chr99a].

The basic idea is to consider any solution with a naked singularity and perturb the initial data $\vartheta = \theta|_{u=0}$ so as to form a trapped sphere. Let $b_\Gamma =: (u^*, v^*)$, then we wish to perturb ϑ only on $[v^*, \infty)$ so that we form no first singularities earlier than v^*. This is illustrated by figure 13.

4.1 Sufficient Criteria for Censorship

There are at least two different ways to perturb initial data so as to guarantee formation of a black hole. These will be discussed separately in §4.1.1 and §4.1.2.

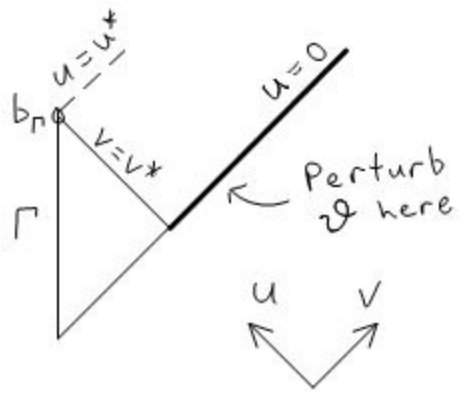


Figure 13: The picture associated to the perturbation technique.

4.1.1 Christodoulou's First Instability Theorem

We first define

$$\sigma := \frac{\mu}{1-\mu}$$

and also the following functions on the ingoing light cone $\{v = v^*\}$ terminating at b_Γ

$$\gamma(u) := \int_0^u \sigma(\tilde{u}, v^*) d\tilde{u},$$

$$I(u) := -\int_0^u e^{-\gamma(\tilde{u})} \zeta(\tilde{u}, v^*) d\tilde{u}.$$

Then, we can state the first instability theorem.

Theorem 4.1 (First Instability Theorem) *Let γ be unbounded, and suppose that*

$$\lim_{u \to u^*} I(u) \neq \vartheta(v^*) \tag{15}$$

or that the limit does not exist. Then A is nonempty (and A terminates at b_Γ).

The proof is by contradiction and relies on Theorem 2.7. Assume for the moment that that Theorem 4.1 is false. Then $R = \mathcal{Q}$ for some solution under the assumptions. So the quantity $\delta(u, v) := \frac{r(u,v)}{r(u,v^*)} - 1$ is a global coordinate, and the region $R^c := \{v \geq v^*\} \cap \{\delta < c\}$ contains no trapped surfaces. Then by Theorem 2.7, it must be the case for some $c \leq 1/e$ and some other $c' \geq 1$ that $\eta \leq c' \delta \log\left(\frac{1}{\delta}\right)$ everywhere on R^c. It turns out that condition (15) is sufficient to violate this bound. (Furthermore, it is possible to replace R^c with

$R^c_\epsilon := [0, u^*] \times [0, \epsilon] \cap \{\delta < c\}$ for any $\epsilon > 0$, thus showing that A terminates at b_Γ.)

The fact that A terminates at b_Γ is interesting, because it shows that $B_0 = b_\Gamma$, but it is irrelevant for the proof of weak cosmic censorship.

At first glance, it may seem that Theorem 4.1 is poorly stated, because BV functions (such as ϑ) may be redefined on a set of measure zero. However, the proof uses an additional assumption that ϑ is upper-semicontinuous, which defines ϑ pointwise.

4.1.2 Christodoulou's Second Instability Theorem

The second instability theorem shows that Theorem 4.1 is not necessary to prove instability. In fact, both instability theorems can be used together to show that the set of solutions featuring a naked singularity has codimension at least 2.

To state the second instability theorem, we first implicitly define $g : (v^*, v^* + \epsilon) \to \mathbb{R}$ by

$$g(v) := e^{-\frac{1}{4}\gamma(u)} \text{ where } v = v^* e^{\frac{2r(u,v^*)}{v^*} e^{-5\gamma(u)}},$$

and also define for any $f : (v^*, v^* + \epsilon) \to \mathbb{R}$,

$$H[f](v) := \sqrt{\frac{1}{\ln(v/v^*)} \int_0^v \left(\frac{v}{v^*} f(v) - f(v^*)\right)^2 d\tilde{v}}.$$

Theorem 4.2 (Second Instability Theorem) *Let γ be unbounded, and suppose that*

$$\lim_{u \to u^*} I(u) = \vartheta(v^*).$$

Then if

$$\limsup_{v \to v^{*+}} \frac{H[\vartheta](v)}{g(v)} = \infty,$$

then the conclusion of Theorem 4.1 holds. In particular, A is nonempty.

4.2 Proof of Instability

To show instability, Christodoulou uses a fact (called the *extension criterion*) from [Chr93] that if γ is bounded as $u \to u^*$, then there is in fact a BV extension beyond b_Γ. So by the definition of b_Γ as a first singularity, it necessarily holds that γ must be unbounded. The use of the extension criterion demonstrates the necessity of a bounded variation formulation for this particular

proof of instability to work. This makes generalisation to stricter regularity classes or beyond spherical symmetry more difficult.

Theorem 4.3 *The set of solutions to the spherically symmetric massless scalar field model exhibiting naked singularities has codimension at least two in BV space.*

Proof Let E be the *exceptional set* of initial data ϑ which lead to naked singularities. We will show that for each $\vartheta \in E$, there is a two-dimensional affine subspace $\Pi_\vartheta \subset BV$ such that $\Pi_\vartheta \cap E = \{\vartheta\}$ and $\Pi_\vartheta \cap \Pi_{\vartheta'} = \emptyset$ whenever $\vartheta \neq \vartheta'$.

We construct a basis $\{f_1, f_2\}$ for $\Pi_\vartheta - \vartheta$ as follows. Let f_1 be a nonnegative function on $[0,\infty)$ where $f_1|_{v<v^*} = 0$, $f_1|_{v\geq v^*}$ is absolutely continuous with $f_1 \to 0$ as $v \to \infty$, and $f_1(v^*) = 1$. Let f_2 be nonnegative absolutely continuous on $[0,\infty)$ such that $f_2|_{v\leq v^*} = 0$, $f_2 \to 0$ as $v \to \infty$, and

$$\limsup_{v\to v^{*+}} \frac{H[f_2](v)}{g(v)} = \infty. \tag{16}$$

The functions f_1 and f_2 generate the perturbations described in figure 13 which are independent of initial data.

For $\vartheta \in E$, define $\Pi_\vartheta := \vartheta + \mathrm{span}\{f_1, f_2\}$ and consider any $\tilde{\vartheta}_{\lambda_1,\lambda_2} \in \Pi_\vartheta$, $\tilde{\vartheta}_{\lambda_1,\lambda_2} := \vartheta + \lambda_1 f_1 + \lambda_2 f_2$. Note that ϑ and $\tilde{\vartheta}_{\lambda_1,\lambda_2}$ coincide on the interval $[0, v^*)$. Therefore, their maximal Cauchy developments coincide on the domain of dependence $[0,u^*) \times [0,v^*) \cap \{v \geq u\}$ of $\{0\} \times [0,v^*)$ and have the same $\gamma(u)$ and $I(u)$ with $\gamma(u)$ diverging by the extension principle for ϑ.

We ask when $\tilde{\vartheta}_{\lambda_1,\lambda_2} \in E$. Since $\vartheta \in E$, by Theorem 4.1, $I \to \vartheta(v^*)$ as $u \to u^*$. But $\tilde{\vartheta}_{\lambda_1,\lambda_2}(v^*) = \vartheta(v^*) + \lambda_1$. So by Theorem 4.1 again, if $\lambda_1 \neq 0$, then $\tilde{\vartheta}_{\lambda_1,\lambda_2} \notin E$. We are left to consider $\tilde{\vartheta}_{0,\lambda_2}$, to which Theorem 4.2 applies since $I \to \vartheta(v^*) = \tilde{\vartheta}_{0,\lambda_2}(v^*)$. But then by choice of f_2, $\limsup_{v\to v^{*+}} H[\tilde{\vartheta}_{0,\lambda_2}](v) = \infty$ whenever $\lambda_2 \neq 0$. So indeed, $\Pi_\vartheta \cap E = \{\vartheta\}$.

We finally ask when $\Pi_\vartheta \cap \Pi_{\vartheta'} \neq \emptyset$. Let $\tilde{\vartheta}_{\lambda_1,\lambda_2} = \tilde{\vartheta}'_{\lambda_1',\lambda_2'}$ for some $\vartheta, \vartheta' \in E$. Once again, since both initial data coincide on $[0,v^*)$, their solutions have the same functions γ and I, with γ diverging by the extension principle. So by Theorem 4.1, we have $\vartheta(v^*) = \lim_{u\to u^*} I = \vartheta'(v^*)$ which means $0 = \tilde{\vartheta}_{\lambda_1,\lambda_2}(v^*) - \tilde{\vartheta}'_{\lambda_1',\lambda_2'}(v^*) = \lambda_1 - \lambda_1'$. Thus, $\vartheta - \vartheta' = (\lambda_2' - \lambda_2) f_2$. But $H[\vartheta - \vartheta']^2 \leq 4\max(H[\vartheta], H[\vartheta'])^2$. This means $(\lambda_2' - \lambda_2) \limsup_{v\to v^{*+}} H[f_2](v)/g(v) < \infty$, which may only be the case if $\lambda_2' = \lambda_2$. So $\Pi_\vartheta \cap \Pi_{\vartheta'} \neq \emptyset$ only if $\vartheta = \vartheta'$. This completes the proof. ∎

Conclusion This essay explored the topic of cosmic censorship by looking at a particular example. Strong cosmic censorship was shown by Christodoulou in [Chr93], but was not discussed in detail in this essay. Weak cosmic censorship was later shown by Christodoulou in [Chr99a] using the technique described in §4. The genericity assumption of both conjectures was made necessary by the non-generic "counterexamples" Christodoulou constructed in [Chr94] which were described in §3.

The story of Christodoulou's work sheds light onto the still open problems that are conjectures 1.2 and 1.3 with weaker assumptions. These are indeed very difficult. Many of Christodoulou's results relied on additional monotonicity, the extension criterion and well-posedness in BV, and simplification of the Einstein equations. It will be necessary to work around some or all of these helpful properties, but I suspect the general approach will more or less be the same. Future proofs of weak cosmic censorship will most likely use a generalised theorem on trapped surface formation which forms the basis for some perturbation of initial data. This will be necessary, because naked singularities have been shown to form at least non-generically.

We have also seen how both weak and strong cosmic censorship are made more understandable by an a priori boundary characterisation like the one given in [Kom11]. This work will be even more valuable as assumptions are relaxed and the picture becomes less clear. While the general cosmic censorship conjectures remain very distant goals, Christodoulou's work is a great first step and provides direction for more steps to follow.

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