A Geometric Method for Constructing A Priori Estimates for Regularly Hyperbolic PDEs

Senior Thesis

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Abstract

This paper is an overview of a geometric technique for constructing a priori estimates for regularly hyperbolic PDEs. A priori estimates are useful for a number of reasons: they provide information about the stability of a solution and provide insights into local and global properties of a solution such as existence, speeds of propagation, and domain of dependence. The technique in this paper is an extension of the Christodoulou–Klainerman vectorfield method, which has led to numerous analytic results. Christodoulou's extension to nonlinear electromagnetism is also discussed at the end of this paper.

1 Introduction

Many PDEs in classical physical theories derive from an action principle. Some examples include electromagnetism, relativistic fluid dynamics, elasticity, and the wave equation. When the action is based on a *regularly hyperbolic* Lagrangian, which will be defined later, it is possible to construct coercive integral identities; this is the main goal of this paper. These identities provide information about the behavior of solutions. This is an extension of Noether's Theorem, and in special cases, these identities are conservation laws. This is the only general method we know of for deriving useful estimates for nonlinear, hyperbolic PDEs.

As outlined in [Spe10b], geometric techniques have led to a number of major results in the past few decades. A few worthy of mention include:

- ⋆ Global nonlinear stability results for the Einstein equations [BZ10], [CK93], [DH06], [KN03], [LR05], [LR10], [RS09], [Spe10a]
- ⋆ Small-data global existence for nonlinear elastic waves [Sid96]

- ⋆ The formation of shocks in solutions to the relativistic Euler equations [Chr07]
- ⋆ Decay results for linear equations on curved backgrounds [AB09], [Blu08], [DR05], [DR08], [DR09], [Hol10a], [Hol10b]
- ⋆ The formation of trapped surfaces in vacuum solutions to the Einstein equations [Chr09], [KR09]
- ⋆ Local existence and non-relativistic limits for the relativistic Euler equations without the use of symmetrizing variables [Spe09a], [Spe09b]

The techniques in this paper concern maps $\phi : \mathbb{R}^m \to \mathbb{R}^n$ which are solutions to a second order PDE derivable from an action principle. The main ideas were extracted from [Chr00]. Let $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ represent the domain and target of a solution to the PDE of interest. Generally, our solution $\phi : M \to N$ will be expressed by its indices $\phi(x) = (\phi^1(x), ..., \phi^n(x)), \quad x \in M$. In this paper, an upper-case Latin index will represent any component (ϕ^A) of ϕ.

We denote by $(x^0, x^1, ..., x^{m-1})$ the standard coordinates of $\mathbb{R}^m$. Lower-case Greek letters will be used for indices corresponding to these coordinates (e.g. $\partial_\mu \phi$). For applications, we reserve x^0 to be the time coordinate and occasionally denote it by t. When it is necessary to distinguish between the time coordinate and the other "spatial" coordinates, an index of 0 will be used to represent the time coordinate and lower-case Latin indices will represent any of the spatial coordinates.

A few standard notations with indices will be used in this paper. Specifically, if an index appears in each term of an equation exactly once, then it shall be understood that the equation holds for every possible value of the index. Furthermore, the Einstein summation convention will be used: if an index appears twice in a term, then it is understood that the term should be summed over each value of the index (e.g. $\partial_\mu y^\mu = \sum_{\mu=0}^{m-1} \partial_\mu y^\mu$).

The method presented in this paper may be summarized as follows. The form of the PDE we shall be studying is

$$h^{\mu\nu}_{AB} \partial_\mu \partial_\nu \phi^B = I_A, \tag{1}$$

where I_A is an inhomogeneous lower order term whose structure is of little significance in this paper. We assume that the PDE (1) is derivable from a Lagrangian $\mathcal{L}[\phi, \partial\phi]$. The derivatives of ϕ, which we shall denote using the variable $\dot{\phi}$, satisfy linearized equations of the form

$$h^{\mu\nu}_{AB} \partial_\mu \partial_\nu \dot{\phi}^B = \dot{I}_A. \tag{2}$$

Up to lower order terms, these equations themselves also derive from a *linearized Lagrangian* $\dot{\mathcal{L}}$, which depends quadratically on $\partial\dot{\phi}$:

$$\dot{\mathcal{L}} := \frac{1}{2} h^{\mu\nu}_{AB} \partial_\mu \dot{\phi}^A \partial_\nu \dot{\phi}^B. \tag{3}$$

We note that the Euler-Lagrange equation for $\dot{\mathcal{L}}$ takes the form

$$\partial_\mu \left(\frac{\partial \dot{\mathcal{L}}}{\partial(\partial_\mu \dot{\phi}^A)} \right) = l.o.t.$$

and so the definition (3) is consistent with (2) up to lower-order terms.

The quantity h, called the *hessian*, is given in terms of the original Lagrangian by

$$h^{\mu\nu}_{AB} := \frac{\partial \mathcal{L}}{\partial_\mu \phi^A \partial_\nu \phi^B}.$$

According to this definition, there is no loss of generality in assuming that the components of the hessian are the same after a simultaneous exchange of upper and lower indices:

$$h^{\mu\nu}_{AB} = h^{\nu\mu}_{BA}.$$

Furthermore, from (1) and the commutativity of partial derivatives, we may without loss of generality assume $h^{\mu\nu}_{AB}$ is symmetric in both upper and lower indices independently:

$$h^{\mu\nu}_{AB} = h^{\nu\mu}_{AB} = h^{\mu\nu}_{BA} = h^{\nu\mu}_{BA}. \tag{4}$$

This fact will be essential in many calculations to follow (especially for the calculations in (20)).

An extremely important quantity associated with the Lagrangian $\dot{\mathcal{L}}$ is the *canonical stress* tensor

$$\begin{aligned} \dot{Q}^\mu{}_\nu &:= \frac{\partial \dot{\mathcal{L}}}{\partial(\partial_\mu \dot{\phi}^A)} \partial_\nu \dot{\phi}^A - \delta^\mu{}_\nu \dot{\mathcal{L}} \\ &= h^{\mu\lambda}_{AB} \partial_\lambda \dot{\phi}^A \partial_\nu \dot{\phi}^B - \frac{1}{2} \delta^\mu{}_\nu h^{\alpha\beta}_{AB} \partial_\alpha \dot{\phi}^A \partial_\beta \dot{\phi}^B. \end{aligned}$$

This quantity is useful, since it depends quadratically on $\partial\dot{\phi}$ and since **its divergence can be expressed only in terms of** $\partial\dot{\phi}$; the PDE (2) can be used to substitute $\partial^2\dot{\phi}$ with $\dot{I}$ (see equation (21)).

For regularly hyperbolic PDEs, the geometric and linear-algebraic properties of h provide special covector-vector pairs (ξ, X), called *coercive pairs*. These coercive pairs are related to special "characteristic directions" of the PDE. Roughly speaking, the *characteristic subsets* $\mathcal{C}^*$ of the cotangent bundle and $\mathcal{C}$ of the tangent bundle each outline an "inner core" (e.g. see Figure 1), which comprises the set of ξ or X which belong to a coercive pair (Theorems 2.6 and 2.8).

The main goal of this paper is to use the divergence theorem to provide useful integral identities. To use the divergence theorem, we need a vectorfield $\vec{J}$. The useful vectorfields are of the form ${}^{(X)}\vec{J} = X \cdot \dot{Q}$ for any vectorfield X belonging to a coercive pair (ξ, X). Integrating the flux of

${}^{(X)}\vec{J}$ over a surface Σ_t with normal covector ξ gives an "energy" quantity $E^2(t)$. The fact that the pair (ξ, X) is coercive makes this quantity useful. It will be shown that $E^2(t)$ controls $\frac{d}{dt}E^2(t)$ (Theorem 3.2). Employing Gronwall's inequality puts an upper bound on $E^2(t)$ based on initial $E^2(t_0)$ for short times.

The outline of this paper is as follows. In Section 2, the regularly hyperbolic Lagrangian and its characteristic subsets $\mathcal{C}^*$ and $\mathcal{C}$ are defined. We then describe how to pick a coercive pair (ξ, X) which is essential for constructing the energy estimate. In some sense, Section 2 is the heart of this paper. Section 3 explains the actual derivation of the energy estimate. Finally, Section 4 summarizes an interesting modification of this technique for application to nonlinear electromagnetism.

This paper is a survey. The techniques in Section 2 are presented in [Chr00] for general differentiable manifolds. Since we restrict our attention to flat Euclidean space, I have done some rearranging and adapted a few of the proofs in hope that they are more accessible to readers without an extensive background in geometry. The technique in Section 3 is also presented in [Chr00], although the way I present it was established through discussions with Dr. Speck. Finally, the technique summarized in Section 4 is employed in [Spe10b], while the general concept is proved in [Chr00].

2 Construction of Coercive Pairs

The purpose of this section is to construct the *coercive pairs* which will be used in the derivation of the energy estimate (Theorem 3.2). At a point $x \in M$, we let T_x^*M denote the cotangent plane and T_xM denote the tangent plane. For a point $q \in N$, we employ the same notation.

Given $\xi \in T_x^*M$, we define a quadratic form $\chi(\xi)$ on T_qN by twice contracting h and ξ:

$$\chi_{AB}(\xi) := h^{\mu\nu}_{AB}\xi_\mu\xi_\nu.$$

We define the *inner cores* of the cotangent and tangent planes as follows.

Definition (*Inner Cores*) Let $\mathcal{I}_x^* \subset T_x^*M$ denote the set of covectors $\xi \in T_x^*M$ for which $\chi(\xi)$ is negative definite. Also, let $\mathcal{J}_x \subset T_xM$ denote the set of vectors X such that for all non-zero ζ satisfying $\zeta(X) = 0$, the quadratic form $\chi(\zeta)$ on T_qN is positive definite. These sets $\mathcal{I}_x^*$ and $\mathcal{J}_x$ are called inner cores. The motivation for this nomenclature will become clear by the end of this section.

Next, we define the *coercive pairs* from elements of the inner cores.

Definition (*Coercive Pair*) We say a pair $(\xi, X) \in T_x^*M \times T_xM$ is *coercive* if $\xi(X) < 0$ and $(\xi, X) \in \mathcal{I}_x^* \times \mathcal{J}_x$. We denote the set of coercive pairs by $\mathcal{T}_x$.

The remainder of this section is devoted to understanding the sets $\mathcal{I}_x^*$, $\mathcal{J}_x$, and $\mathcal{T}_x$. We will assume that these sets are non-empty, which is to say the Lagrangian is *regularly hyperbolic* according to the following definition.

Definition (*Regular Hyperbolicity*) A Lagrangian $\mathcal{L}$ or its associated hessian h is said to be regularly hyperbolic at $x \in M$ if and only if the corresponding set $\mathcal{T}_x$ is non-empty. This is to say that there exists a coercive pair (ξ, X).

Proposition 2.1 *Let $\xi \in \mathcal{I}_x^*$ and $X \in \mathcal{J}_x$. Then $\xi(X) \neq 0$. Furthermore, if $\lambda \neq 0$ is a scalar, then $\lambda X \in \mathcal{J}_x$ and $\lambda\xi \in \mathcal{I}_x^*$. Finally, if $\xi(X) > 0$ then $(\xi, -X)$ and $(-\xi, X)$ are both coercive pairs.*

Proof If $X \in \mathcal{J}_x$ and $\xi(X) = 0$, then $h^{\mu\nu}_{AB}\xi_\mu\xi_\nu$ is necessarily positive definite. If $\xi \in \mathcal{I}_x^*$, then $h^{\mu\nu}_{AB}\xi_\mu\xi_\nu$ is necessarily negative definite. These two possibilities are mutually exclusive, so $\xi(X) \neq 0$.

For any nonzero scalar λ, we have $h^{\mu\nu}_{AB}(\lambda\xi)_\mu(\lambda\xi)_\nu = \lambda^2 h^{\mu\nu}_{AB}\xi_\mu\xi_\nu$. It follows that this quadratic form never changes sign after applying $\xi \mapsto \lambda\xi$, which means $\lambda\xi \in \mathcal{I}_x^*$. Also, the set of $\zeta \in T_x^*M$ which are orthogonal to λX is precisely the same set corresponding to X. Therefore, $\lambda X \in \mathcal{J}_x$.

If $\xi(X) > 0$, then $(-\xi)(X) = \xi(-X) < 0$. Letting $\lambda = -1$, we see that $-\xi \in \mathcal{I}_x^*$ and $-X \in \mathcal{J}_x$. Therefore, $(-\xi, X)$ and $(\xi, -X)$ belong to $\mathcal{T}_x$. ∎

These basic facts provide insight into the shape of the sets $\mathcal{I}_x^*$ and $\mathcal{J}_x$. Specifically, if T_x^*M is identified with $\mathbb{R}^m$, then $\mathcal{I}_x^*$ appears as a union of rays emanating from the origin, and if one of these rays belongs to the set, then so does its opposite. The same can be said of $\mathcal{J}_x$. For a wave equation, the sets $\mathcal{I}_x^*$ and $\mathcal{J}_x$ are the interiors of the null cones in T_x^*M and T_xM.

Let $L(T_xM, T_qN)$ denote the set of linear functions $V : T_xM \to T_qN$. We denote the components of an element $V \in L(T_xM, T_qN)$ by V_μ^A. Note that $\partial_\mu\dot{\phi}^A \in L(T_xM, T_qN)$ is such an element. Given a pair (ξ, X), the quantity $\dot{Q}^\mu{}_\nu\xi_\mu X^\nu$ is quadratic in the variable $\partial\dot{\phi}$. This is a special case of an important bilinear form acting on $L(T_xM, T_qN)$ called the *Noether Transform*.

Definition (*Noether Transform*) Let $\xi \in T_x^*M$ and $X \in T_xM$. Then the Noether transform of h with respect to ξ and X is the bilinear form $\mathcal{N}(\xi, X)[\cdot,\cdot]$ acting on $L(T_xM, T_qN)$ given by

$$\begin{aligned}\mathcal{N}(\xi, X)[V_1, V_2] &:= h(\xi \otimes V_1(X), V_2) + h(V_1, \xi \otimes V_2(X)) - h(V_1, V_2)\xi(X) \\ &= (h^{\lambda\nu}_{AB}X^\mu + h^{\mu\lambda}_{AB}X^\nu - h^{\mu\nu}_{AB}X^\lambda)\xi_\lambda {V_1}_\mu^A {V_2}_\nu^B. \qquad (5)\end{aligned}$$

Remark According to this definition, $\dot{Q}^\mu{}_\nu X^\nu\xi_\mu = \frac{1}{2}\mathcal{N}(\xi, X)[\partial\dot{\phi}, \partial\dot{\phi}]$.

Of special interest is the way $\mathcal{N}(\xi, X)$ behaves on the set

$$R_\xi := \{\xi \otimes P + \zeta \otimes Q : \zeta \in T_x^* M, P, Q \in T_q N\}.$$

This set is of special interest, because it is encountered in the proof of Lemma 3.1. Assuming $\xi(X) \neq 0$, there is a decomposition

$$R_\xi = L_\xi \oplus \Sigma_X \tag{6}$$

where

$$L_\xi = \{\xi \otimes P : P \in T_q N\} \text{ and } \Sigma_X = \{\zeta \otimes Q : \zeta(X) = 0, Q \in T_q N\}.$$

Indeed, let $V = \xi \otimes P + \zeta \otimes Q$. Then $P' = P + \frac{\zeta(X)}{\xi(X)} Q$ and $\zeta' = \zeta - \frac{\zeta(X)}{\xi(X)} \xi$ satisfy $V = \xi \otimes P' + \zeta' \otimes Q$ and $\zeta'(X) = 0$.

It is helpful to calculate $\mathcal{N}(\xi, X)[V, V]$ for an arbitrary element $V \in R_\xi$. In light of the decomposition (6), we let $V = \xi \otimes P + \zeta \otimes Q$ where $\zeta(X) = 0$. We calculate $\mathcal{N}(\xi, X)[V, V]$ in three parts:

$$\begin{aligned}
\mathcal{N}(\xi, X)[\xi \otimes P, \xi \otimes P] &= (h^{\lambda\nu}_{AB} X^\mu + h^{\mu\lambda}_{AB} X^\nu - h^{\mu\nu}_{AB} X^\lambda) \xi_\lambda \xi_\mu \xi_\nu P^A P^B \\
&= h^{\mu\nu}_{AB} \xi_\mu \xi_\nu P^A P^B \xi(X)
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(\xi, X)[\xi \otimes P, \zeta \otimes Q] &= (h^{\lambda\nu}_{AB} X^\mu + h^{\mu\lambda}_{AB} X^\nu - h^{\mu\nu}_{AB} X^\lambda) \xi_\lambda \xi_\mu \zeta_\nu P^A Q^B \\
&= h^{\mu\nu}_{AB} \xi_\mu \xi_\nu P^A Q^B \zeta(X) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(\xi, X)[\zeta \otimes Q, \zeta \otimes Q] &= (h^{\lambda\nu}_{AB} X^\mu + h^{\mu\lambda}_{AB} X^\nu - h^{\mu\nu}_{AB} X^\lambda) \xi_\lambda \zeta_\mu \zeta_\nu Q^A Q^B \\
&= 2 h^{\mu\nu}_{AB} \xi_\mu \zeta_\nu Q^A Q^B \zeta(X) - h^{\mu\nu}_{AB} \zeta_\mu \zeta_\nu Q^A Q^B \xi(X) \\
&= -h^{\mu\nu}_{AB} \zeta_\mu \zeta_\nu Q^A Q^B \xi(X).
\end{aligned}$$

From the bilinearity of $\mathcal{N}(\xi, X)[\cdot, \cdot]$ and the three calculations above, we have the following general formula.

$$\mathcal{N}(\xi, X)[V, V] = \xi(X) \left(h^{\mu\nu}_{AB} \xi_\mu \xi_\nu P^A P^B - h^{\mu\nu}_{AB} \zeta_\mu \zeta_\nu Q^A Q^B \right) \tag{8}$$

This formula gives rise to an important relation between the Noether transform and $\mathcal{T}_x$. It basically says that the Noether transform associated to any coercive pair is positive definite on the set R_ξ.

Proposition 2.2 *Let $\mathcal{N}$ correspond to a regularly hyperbolic Lagrangian. The following two statements hold.*
(a) $X \in \mathcal{J}_x \Leftrightarrow$ There exists $\xi \in \mathcal{I}_x^*$ for which $\xi(X) < 0$ and $\mathcal{N}(\xi, X)$ is positive definite on R_ξ.
(b) $\xi \in \mathcal{I}_x^* \Leftrightarrow$ There exists $X \in \mathcal{J}_x$ for which $\xi(X) < 0$ and $\mathcal{N}(\xi, X)$ is positive definite on R_ξ.

Proof We first consider the backward directions. Let ξ satisfy the assumptions in (a$\Leftarrow$). According to (8), since $\xi(X) < 0$, the requirement that $\mathcal{N}(\xi, X)$ be positive definite on R_ξ stipulates $h^{\mu\nu}_{AB}\zeta_\mu\zeta_\nu Q^A Q^B > 0$ for arbitrary nonzero Q and ζ satisfying $\zeta(X) = 0$ (since $\zeta \otimes Q \in R_\xi$). This proves (a$\Leftarrow$). A similar argument shows the backward direction (b$\Leftarrow$).

Now, consider the forward direction (a$\Rightarrow$). Assume that $h^{\mu\nu}_{AB}\zeta_\mu\zeta_\nu$ is positive definite for all non-zero ζ satisfying $\zeta(X) = 0$. Since we restrict our attention to regularly hyperbolic Lagrangians, there exists $\xi \in \mathcal{I}^*_x$. $h^{\mu\nu}_{AB}\xi_\mu\xi_\nu$ is negative definite, so $\xi(X) \neq 0$. This allows us to pick $\xi' = \pm\xi$ where the sign is chosen so that $\xi'(X) < 0$. Notice that $h^{\mu\nu}_{AB}\xi'_\mu\xi'_\nu = h^{\mu\nu}_{AB}\xi_\mu\xi_\nu$ is necessarily negative definite. Given our assumption, we see again from (8) that $\mathcal{N}(\xi', X)$ is positive definite on $R_{\xi'}$. The proof of (b$\Rightarrow$) is similar to this argument. ∎

The main point to take from Proposition 2.2 is the following Corollary.

Corollary 2.3 *Let $\mathcal{N}(\xi, X)$ be the Noether transform of a regularly hyperbolic hessian h. Assume that $\xi(X) < 0$. Then (ξ, X) is a coercive pair if and only if $\mathcal{N}(\xi, X)$ is positive definite on R_ξ.*

Proof The "only if" direction follows from equation (8) and the "if" direction follows from Proposition 2.2. ∎

In fact, Christodoulou defines the set $\mathcal{T}_x$ this way in [Chr00].

Let $Null(\chi(\xi)) \subset T_qN$ denote the null space of the quadratic form $\chi(\xi)$ introduced earlier.

$$Null(\chi(\xi)) := \{P \in T_qN : \chi(\xi) \cdot P = 0\}$$

From elementary linear algebra, this null space is non-trivial (i.e. $Null(\chi(\xi)) \neq \{0\}$) whenever $\det(\chi(\xi)) = 0$. The set of ξ for which this is the case is the *characteristic subset of the cotangent space* of $x \in M$.

Definition (*Characteristic subset $\mathcal{C}^*_x$*) We define the *characteristic subset of the cotangent space* of $x \in M$ by $\mathcal{C}^*_x := \{\xi \in T^*_xM : Null(\chi(\xi)) \neq \{0\}\}$.

For $P \in T_qN$, we define a bilinear form $\psi(P)$ on T^*_xM by

$$\psi(P)[\xi, \zeta] := h(\xi \otimes P, \zeta \otimes P). \tag{9}$$

In component form, $\psi^{\mu\nu}(P) = h^{\mu\nu}_{AB}P^A P^B$. For each $\xi \in \mathcal{C}^*_x$ we define a subset of T_xM by

$$\Lambda(\xi) := \{\psi(P) \cdot \xi : P \neq 0 \in Null(\chi(\xi))\}.$$

Now, we define the *characteristic subset of the tangent space* of $x \in M$.

Definition (*Characteristic subset* $\mathcal{C}_x$) We define the *characteristic subset of the tangent space* of $x \in M$ by

$$\mathcal{C}_x := \bigcup_{\xi \in \mathcal{C}_x^*} \Lambda(\xi).$$

Remark If $n = 1$, then

$$\mathcal{C}_x^* = \{\xi : h^{\mu\nu}\xi_\mu\xi_\nu = 0\}$$

and

$$\mathcal{C}_x = \{Y : Y^\mu = h^{\mu\nu}\xi_\nu, \xi \in \mathcal{C}_x^*\}.$$

The remainder of this section is devoted to understanding the relation between $\mathcal{C}_x^*, \mathcal{C}_x$ and $\mathcal{I}_x^*, \mathcal{J}_x$. This relation is summarized at the end of this section and an example picture (Figure 1) is provided.

Proposition 2.4 $\mathcal{C}_x^* \cap \mathcal{I}_x^* = \varnothing$ *and likewise,* $\mathcal{C}_x \cap \mathcal{J}_x = \varnothing$.

Proof If $\xi \in \mathcal{C}_x^*$, there is some nonzero $Q \in T_qN$ for which $\chi(\xi) \cdot Q = 0$. It trivially follows that $\chi(\xi)[Q, Q] = 0$, which means $\chi(\xi)$ is not negative definite (i.e. $\xi \notin \mathcal{I}_x^*$).

Likewise, if $X \in \mathcal{C}_x$, then we may write $X^\mu = h^{\mu\nu}_{AB}\xi_\nu Q^A Q^B$ for some $\xi \in \mathcal{C}_x^*$ and $Q \in Null(\chi(\xi))$. According to this expression for X^μ, we have $\xi(X) = X^\mu\xi_\mu = h^{\mu\nu}_{AB}\xi_\mu\xi_\nu Q^A Q^B = 0$. Then if $X \in \mathcal{J}_x$, we require $\chi(\xi)$ to be positive definite. But $\chi(\xi)[Q, Q] = h^{\mu\nu}_{AB}\xi_\mu\xi_\nu Q^A Q^B = 0$. ∎

Proposition 2.5 *The sets* $\mathcal{I}_x^*$ *and* $\mathcal{J}_x$ *are open.*

Proof Given a quadratic form $Q[\cdot,\cdot]$ over a finite dimensional vector space V endowed with an inner product $\cdot$ and corresponding norm $\| \quad \|$, we define $[Q]_{\inf} = \inf_{v \in V} \frac{Q[v,v]}{v \cdot v}$ and $[Q]_{\sup} = \sup_{v \in V} \frac{Q[v,v]}{v \cdot v}$. In finite dimensions, the condition of negative definiteness is equivalent to $[Q]_{\sup} < 0$.

Endow T_qN with the standard Euclidean inner product and norm. Supposing $\xi \in \mathcal{I}_x^*$, then we let $\xi' = \xi + \epsilon\hat{\zeta}$ where $\hat{\zeta}$ is a unit covector.

$$\begin{aligned}
\left[h^{\mu\nu}_{AB}\xi'_\mu\xi'_\nu\right]_{\sup} &= \sup_{P \in T_qN} h^{\mu\nu}_{AB}(\xi + \epsilon\hat{\zeta})_\mu(\xi + \epsilon\hat{\zeta})_\nu \frac{P^A P^B}{\|P\|^2} \\
&= \sup_{P \in T_qN} \left(h^{\mu\nu}_{AB}\xi_\mu\xi_\nu + 2\epsilon h^{\mu\nu}_{AB}\xi_\mu\hat{\zeta}_\nu + \epsilon^2 h^{\mu\nu}_{AB}\hat{\zeta}_\mu\hat{\zeta}_\nu\right) \frac{P^A P^B}{\|P\|^2} \\
&\leq \left[h^{\mu\nu}_{AB}\xi_\mu\xi_\nu\right]_{\sup} + 2\epsilon\left[h^{\mu\nu}_{AB}\xi_\mu\hat{\zeta}_\nu\right]_{\sup} + \epsilon^2\left[h^{\mu\nu}_{AB}\hat{\zeta}_\mu\hat{\zeta}_\nu\right]_{\sup} \qquad (10)
\end{aligned}$$

The terms $[h^{\mu\nu}_{AB}\xi_\mu\hat{\zeta}_\nu]_{\sup}$ and $[h^{\mu\nu}_{AB}\hat{\zeta}_\mu\hat{\zeta}_\nu]_{\sup}$ are continuous in $\hat{\zeta}$ and defined on the unit sphere in T_x^*M, which means they are each bounded by some maximum. Since $\xi \in \mathcal{I}_x^*$, the first term in (10) is negative, so we can pick

$\epsilon_0 > 0$ so that whenever $0 \le \epsilon < \epsilon_0$, (10) remains negative regardless of $\hat{\zeta}$. This means $\xi' \in \mathcal{I}_x^*$ for all $0 \le \epsilon < \epsilon_0$ and all $\hat{\zeta}$. Therefore the neighborhood of radius ϵ_0 about ξ belongs to $\mathcal{I}_x^*$.

We turn our attention to the set $\mathcal{J}_x$. Let $X \in \mathcal{J}_x$. By Proposition 2.2, there is some $\xi \in \mathcal{I}_x^*$ so that $\xi(X) < 0$ and $\mathcal{N}(\xi, X)$ is a positive definite quadratic form on R_ξ. Since $\xi(\cdot)$ is a continuous function, there is some neighborhood $N'(X)$ about X for which any vector $X' \in N'(X)$ satisfies $\xi(X') < 0$. In light of Proposition 2.2, it suffices to find a smaller neighborhood $N''(X) \subset N'(X)$ such that $\mathcal{N}(\xi, X'')$ is positive definite on R_ξ for each $X'' \in N''(X)$. This is possible, because $\mathcal{N}(\xi, X)$ is linear in X and the set $\{\hat{V} \in R_\xi : \|\hat{V}\| = 1\}$ is compact according to the norm defined in the following paragraph.

We introduce a norm on $L(T_xM, T_qN)$ in order to show that the subset of linear maps in R_ξ with unit norm is compact. Endow T_x^*M with the standard Euclidean norm and inner product, and in doing so, identify $L(T_xM, T_qN)$ with $L(T_x^*M, T_qN)$. We have the following decomposition

$$L(T_x^*M, T_qN) = W_1 \oplus W_2,$$

where W_2 is the subspace with dimension $n(m-1)$ of linear maps with null space containing ξ, and W_1 is the subspace with dimension n of linear maps of the form $\xi \otimes P$, $P \in T_qN$. We separately define norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the spaces W_1 and W_2. Endow T_qN with the standard Euclidean norm. For $T_i \in W_i$, define $\|T_i\|_i := \sup_{\|\zeta\|=1} \|T_i\zeta\|$. The rank 1 elements of W_2 each have an image with dimension 1, while the remaining elements of W_2 have images which contain at least two linearly independent elements of T_qN. Given any linearly independent pair of elements in T_qN which correspond to unit covectors in T_x^*M, there is a distance $\epsilon > 0$ such that no one dimensional image can be within ϵ of both members of the pair. Thus, the set of rank 1 elements of W_2 is closed under $\|\cdot\|_2$ and so the set R_ξ is closed under the norm $\|(T_{W_1}, T_{W_2})\| := \max_i \|T_{W_i}\|_i$ defined on $L(T_x^*M, T_qN)$. Identifying $L(T_x^*M, T_qN)$ with $\mathbb{R}^{mn}$, the set $\{\hat{V} \in R_\xi : \|\hat{V}\| = 1\}$ is both closed and bounded, and is therefore compact. ∎

The following two Theorems concern the boundaries $\partial\mathcal{I}_x^*$ and $\partial\mathcal{J}_x$ which, since $\mathcal{I}_x^*$ and $\mathcal{J}_x$ are open, are the sets of limit points of sequences in $\mathcal{I}_x^*$ or $\mathcal{J}_x$ which do not belong to $\mathcal{I}_x^*$ or $\mathcal{J}_x$.

Theorem 2.6 $\partial\mathcal{I}_x^* \subset \mathcal{C}_x^*$

Proof Let $\xi \in \partial\mathcal{I}_x^*$. We can approach ξ by some sequence $\{\xi_n\} \subset \mathcal{I}_x^*$ and observe that for any nonzero $P \in T_qN$, we have $\chi(\xi_n)[P, P] < 0$. Since $\chi(\cdot)$ is continuous, then $\chi(\xi)[P, P] \le 0$. From Proposition 2.5, we know that

$\xi \notin \mathcal{I}_x^*$, which implies $\forall P \in T_qN, \chi(\xi)[P,P] \leq 0$ and $\chi(\xi)[P_0,P_0] = 0$ for some $P_0 \neq 0$. As a result, for any arbitrary $\dot{P} \in T_qN$,

$$\begin{aligned} 0 \geq & \chi(\xi)(P_0 + \epsilon\dot{P}, P_0 + \epsilon\dot{P}) \qquad (11) \\ &= \chi(\xi)[P_0,P_0] + 2\epsilon\chi(\xi)[P_0,\dot{P}] + \epsilon^2\chi(\xi)[\dot{P},\dot{P}] \\ &= \epsilon\left(2\chi(\xi)[P_0,\dot{P}] + \epsilon\chi(\xi)[\dot{P},\dot{P}]\right). \qquad (12) \end{aligned}$$

We may set $\epsilon > 0$ arbitrarily small and may reverse the sign of $\dot{P}$ so as to change the sign of the first term in (12). The only way that (11) will still hold is if $\chi(\xi)[P_0,\dot{P}] = 0$ for all $\dot{P}$. This means $h^{\mu\nu}_{AB}\xi_\mu\xi_\nu P_0^A = 0$, which means $\xi \in \mathcal{C}_x^*$. ∎

To prove the analogous theorem for the tangent space, we first observe the following lemma.

Lemma 2.7 *Let h be regularly hyperbolic. Then for all $Q \neq 0 \in T_qN$, the matrix $\psi(Q)$ is invertible.*

Proof Let (ξ, X) be a coercive pair. Then $\xi(X) \neq 0$. For any covector $\eta \in T_x^*M$, we have

$$\eta = \frac{\eta(X)}{\xi(X)}\xi + \left(\eta - \frac{\eta(X)}{\xi(X)}\xi\right).$$

Thus, we have a decomposition for T_x^*M

$$T_x^*M = \mathrm{span}(\xi) \oplus \{\zeta : \zeta(X) = 0\}. \qquad (13)$$

Since both $\xi \otimes Q$ and $\zeta \otimes Q$ ($\zeta(X) = 0$) belong to R_ξ, we have $\mathcal{N}(\xi,X)[\xi \otimes Q, \xi \otimes Q] > 0$ and $\mathcal{N}(\xi,X)[\zeta \otimes Q, \zeta \otimes Q] > 0$. As a result of (8), it follows that $\psi(Q)$ is negative definite on $\mathrm{span}(\xi)$ and positive definite on $\{\zeta : \zeta(X) = 0\}$. In light of the decomposition (13), $\psi(Q)$ is invertible. ∎

Theorem 2.8 $\partial\mathcal{J}_x \subset \mathcal{C}_x$

Proof This proof is similar to the proof of Proposition 2.6, and so some repeated steps will not be explicitly stated. Let $X \in \partial\mathcal{J}_x$ and pick a sequence $X_n \to X$ in $\mathcal{J}_x$. According to Proposition 2.2, we pick $\xi_1 \in \mathcal{I}_x^*$ corresponding to X_1 so that $\xi_1(X_1) < 0$ and $\mathcal{N}(X_1,\xi_1)$ is positive definite on R_{ξ_1}. We now choose a new ξ depending on the conditions of $\xi_1(X)$. If $\xi_1(X) < 0$, then we let $\xi = \xi_1$. If $\xi_1(X) > 0$, then we let $\xi = -\xi_1$ so that $\xi(X) < 0$. Lastly, if $\xi_1(X) = 0$, then we pick a nearby $\xi \in \mathcal{I}_x^*$ so that $\xi(X) < 0$. We know this is possible, because the set $\{\xi : \xi(X) = 0\}$ is a plane of dimension $m-1$ and $\mathcal{I}_x^*$ is open.

Given this new $\xi \in \mathcal{I}_x^*$, since $\xi(X) < 0$, there exists an integer N for which $n > N$ implies $\xi(X_n) < 0$. By Corollary 2.3, on the subsequence $\{X_{n>N}\}$,

the quadratic form $\mathcal{N}(X_n, \xi)$ is positive definite on R_ξ. Furthermore, $\mathcal{J}_x$ is open and $\mathcal{N}(\xi, \cdot)$ is linear, which implies $\inf_{V \in R_\xi} \frac{\mathcal{N}(\xi,X)[V,V]}{V \cdot V} \geq 0$.

By the same logic as in Theorem 2.6, there is a nonzero $V_0 \in R_\xi$ (given by $V_0 = \xi \otimes P_0 + \zeta_0 \otimes Q_0$ where $\zeta_0(X) = 0$) which obtains $\mathcal{N}(\xi, X)[V_0, V_0] = 0$. It follows for any variation $\dot{V} \in R_\xi$ that

$$\mathcal{N}(\xi, X)[V_0, \dot{V}] = 0. \tag{14}$$

We first consider (14) when the variation is of the form $\dot{V} = \xi \otimes \dot{P}$. Using (5), we calculate

$$\begin{aligned} 0 &= \mathcal{N}^{\mu\nu}_{AB} V_{0\mu}{}^A \dot{P}^B \xi_\nu \\ &\Downarrow \\ 0 &= \mathcal{N}^{\mu\nu}_{AB} V_{0\mu}{}^A \xi_\nu \\ &= h^{\lambda\nu}_{AB} \xi_\lambda \xi_\nu X^\mu V_{0\mu}{}^A \\ &= \chi_{AB}(\xi) X^\mu V_{0\mu}{}^A \\ &= \chi_{AB}(\xi) \xi(X) P_0^A. \end{aligned}$$

Since $\xi \in \mathcal{I}_x^*$, the matrix $\chi_{AB}(\xi)$ is non-degenerate, so $P_0 = 0$. Thus, we have $V_0 = \zeta_0 \otimes Q_0 \neq 0$. Considering now (14) in the case $\dot{V} = \zeta_0 \otimes \dot{Q}$, we calculate

$$\begin{aligned} 0 &= \mathcal{N}^{\mu\nu}_{AB} \zeta_{0\mu} Q_0^A \zeta_{0\nu} \\ &\Downarrow (\zeta_0(X) = 0) \\ 0 &= h^{\mu\nu}_{AB} \zeta_{0\mu} \zeta_{0\nu} Q_0^A = \chi_{AB}(\zeta_0) Q_0^A. \end{aligned}$$

By definition, we have that $\zeta_0 \in \mathcal{C}_x^*$ and $Q_0 \in Null(\chi(\zeta_0))$. Lastly, we consider (14) in the case $\dot{V} = \dot{\zeta} \otimes Q_0$:

$$\begin{aligned} 0 &= \mathcal{N}^{\mu\nu}_{AB} V_{0\mu}{}^A Q_0^B \\ &= \mathcal{N}^{\mu\nu}_{AB} Q_0^A Q_0^B \zeta_{0\mu} \\ &= h^{\mu\lambda}_{AB} X^\nu \xi_\lambda Q_0^A Q_0^B \zeta_{0\mu} - h^{\mu\nu}_{AB} X^\lambda \xi_\lambda Q_0^A Q_0^B \zeta_{0\mu} \\ &= (h^{\mu\lambda}_{AB} \xi_\lambda Q_0^A Q_0^B \zeta_{0\mu}) X^\nu - \xi(X)(\psi^{\mu\nu}(Q_0)\zeta_{0\mu}). \end{aligned} \tag{15}$$

According to Lemma 2.7, both terms in (15) are nonzero. For any scalar k, we have $\zeta_0 \otimes Q_0 = (k\zeta_0) \otimes (\frac{1}{k} Q_0)$, so by scaling Q appropriately we may assume without loss of generality that $h^{\mu\lambda}_{AB} \xi_\lambda Q_0^A Q_0^B \zeta_{0\mu} = \xi(X)$ since the expression $h^{\mu\lambda}_{AB} \xi_\lambda Q_0^A Q_0^B \zeta_{0\mu}$ is quadratic in Q_0 but linear in ζ_0. In this case, (15) becomes

$$X = \psi(Q_0) \cdot \zeta_0$$

Since $Q_0 \in Null(\chi(\zeta_0))$, this means $X \in \Lambda(\zeta_0) \subset \mathcal{C}^*$. ∎

Definition Fix $X \in \mathcal{J}_x$ and $\xi \in \mathcal{I}_x^*$. Then we define the sets

$$\mathcal{J}_x^+ := \{Y \in \mathcal{J}_x : \xi(Y) > 0\} \text{ and } \mathcal{J}_x^- := \{Y \in \mathcal{J}_x : \xi(Y) < 0\}.$$

Likewise, we define

$$\mathcal{I}_x^{*+} := \{\zeta \in \mathcal{I}_x^* : \zeta(X) > 0\} \text{ and } \mathcal{I}_x^{*-} := \{\zeta \in \mathcal{I}_x^* : \zeta(X) < 0\}.$$

Remark It follows from Proposition 2.1 that $\mathcal{J}_x = \mathcal{J}_x^+ \cup \mathcal{J}_x^-$ and $\mathcal{I}_x^* = \mathcal{I}_x^{*+} \cup \mathcal{I}_x^{*-}$ and that $\mathcal{J}_x^- = -\mathcal{J}_x^+$ and $\mathcal{I}_x^{*-} = -\mathcal{I}_x^{*+}$.

Proposition 2.9 *The sets $\mathcal{J}_x^+$, $\mathcal{J}_x^-$, $\mathcal{I}_x^{*+}$, and $\mathcal{I}_x^{*-}$ are convex.*

Proof In light of the above remark, it suffices only to consider $\mathcal{J}_x^-$ and $\mathcal{I}_x^{*-}$. We first show that $\mathcal{J}_x^-$ is convex. For a fixed $\xi \in \mathcal{I}_x^*$, let $X_0, X_1 \in \mathcal{J}_x^-$ so that $\xi(X_i) < 0$. Then both quadratic forms $\mathcal{N}(\xi, X_i)$ are positive definite on R_ξ. Let $X_t = tX_1 + (1-t)X_0$ for any $t \in [0,1]$. Since $\mathcal{N}(\xi, \cdot)$ is linear, the quadratic form $\mathcal{N}(\xi, X_t) = (1-t)\mathcal{N}(\xi, X_0) + t\mathcal{N}(\xi, X_1)$ is also positive definite on R_ξ. And since $\xi(X_t) = (1-t)\xi(X_0) + t\xi(X_1) < 0$, we see that $X_t \in \mathcal{J}_x^-$ by Proposition 2.2.

Now, we show that $\mathcal{I}_x^{*-}$ is convex. For a fixed $X \in \mathcal{J}_x$, let $\xi_0, \xi_1 \in \mathcal{I}_x^{*-}$ so that $\xi_i(X) < 0$. We first observe that $\xi_t(X) = t\xi_1(X) + (1-t)\xi_0(X) < 0$ for all $t \in [0,1]$, so it is only necessary to show that $\xi_t \in \mathcal{I}_x^*$. If this is not the case, there is some t_0 for which $\xi_{t_0} \in \partial\mathcal{I}_x^* \subseteq \mathcal{C}_x^*$. So we select $P \neq 0 \in Null(\chi(\xi_{t_0}))$ and conclude from (8) that $\mathcal{N}(\xi_{t_0}, X)[\xi_{t_0} \otimes P, \xi_{t_0} \otimes P] = 0$. But since all rank-one elements belong to R_{ξ_i}, we have that $\xi_{t_0} \otimes P \in R_{\xi_i}$. Thus, $\mathcal{N}(\xi_{t_0}, X)[\xi_{t_0} \otimes P, \xi_{t_0} \otimes P] = (t_0\mathcal{N}(\xi_1, X) + (1-t_0)\mathcal{N}(\xi_0, X))[\xi_{t_0} \otimes P, \xi_{t_0} \otimes P] > 0$. This is a contradiction. ∎

Summary The sets $\mathcal{I}_x^*$ and $\mathcal{J}_x$ are precisely the sets of $\xi \in T_x^*M$ and $X \in T_xM$ for which the Noether transform $\mathcal{N}(\xi, X)$ is positive definite on R_ξ whenever $\xi(X) < 0$. There are three key properties that help us to determine these sets: they do not intersect with the respective characteristic sets, their boundaries belong to the respective characteristic sets, and they take the form of a union of two opposite convex sets. The following example along with Figure 1 demonstrates the usefulness of these properties.

Example Let $\dim(N) = 2$ and

$$\begin{aligned} h_{11}^{\mu\nu} &= \text{diag}(-1, 1, ..., 1) \\ h_{22} &= \text{diag}(-\frac{1}{4}, 1, ..., 1) \\ h_{12} &= h_{21} = 0. \end{aligned}$$

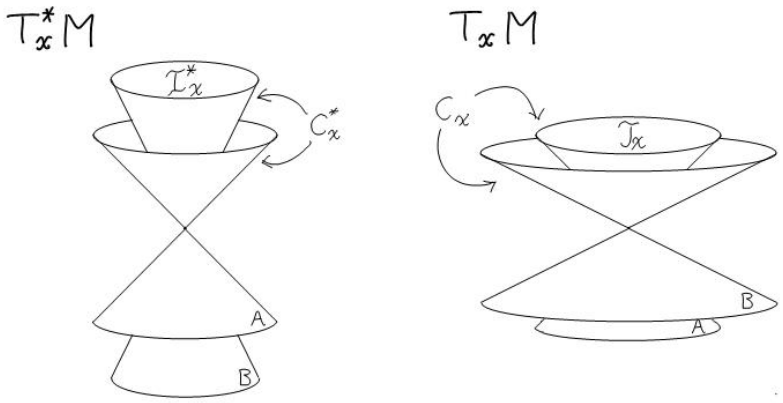


Figure 1: A diagram of the characteristic subsets and the inner cores $\mathcal{I}_x^*$ and $\mathcal{J}_x$ from the example

Assuming h is regularly hyperbolic, we seek $\mathcal{I}_x^*$ and $\mathcal{J}_x$ for the "decoupled generalized wave equation" $h_{AB}^{\mu\nu}\partial_\mu\partial_\nu\phi^A = 0$. To do this, we first determine $\mathcal{C}_x^*$ by setting $\det(h_{AB}^{\mu\nu}\xi_\mu\xi_\nu) = 0$. This happens when either $h_{11}^{\mu\nu}\xi_\mu\xi_\nu = 0$ or $h_{22}^{\mu\nu}\xi_\mu\xi_\nu = 0$. And so

$$\mathcal{C}_x^* = \{|\xi_0|^2 = \sum_{i=1}^{m-1}|\xi_i|^2\} \cup \{\frac{1}{4}|\xi_0|^2 = \sum_{i=1}^{m-1}|\xi_i|^2\}$$

This set is shown on the left in Figure 1 where the outer cone (A) and the inner cone (B) correspond to h_{11} and h_{22} respectively. There are five open, connected regions in T_x^*M which have a boundary belonging to $\mathcal{C}_x^*$ and do not intersect $\mathcal{C}_x^*$. Three of them are not convex and therefore $\mathcal{I}_x^*$ is the "inner core" of the inner cone (B) given by $\{\frac{1}{4}|\xi_0|^2 > \sum_{i=1}^{m-1}|\xi_i|^2\}$.

For any ξ belonging to the cone (A), the null space $\text{Null}(\chi(\xi))$ is the set $\{(\phi^1, 0)\}$. It follows that $\Lambda(\xi) = \{|\phi^1|^2 h_{11}^{\mu\nu}\xi_\mu : \phi^1 \neq 0\}$ is a ray in the cone (A) in the tangent space on the right of Figure 1. A similar calculation follows for ξ belonging to cone (B). The entire set $\mathcal{C}_x$ is the union of these two cones. Recalling the properties of $\mathcal{J}_x$, we determine that $\mathcal{J}_x$ must be the "inner core" of the inner cone (A) given by $\{|X^0|^2 > \sum_{i=1}^{m-1}|X^i|^2\}$. So in this example, both cones (A) and (B) determine the set $\mathcal{T}_x$.

3 Deriving the Energy Estimate

3.1 Integrated-Coerciveness Properties of Currents

In order to introduce the main theorem and motivation for coercive pairs, it is necessary to introduce geometrically motivated foliations. Let $\Omega \subset M$ be a m-dimensional domain, which we shall endow with the standard

Euclidean metric e. We need the metric e in order to apply the divergence theorem. We will assume the existence of a well-behaved time function on Ω, which we will now define.

Definition We define a *time function* to be a differentiable mapping $f : \Omega \to [t_0, t_1]$ such that $\nabla f|_x \in \mathcal{I}_x^*$ for all $x \in \Omega$ and such that each set $\Sigma_t := f^{-1}(t)$ is a connected $m-1$ dimensional surface. We say that f is *well-behaved* if $|\nabla f| > \epsilon$ everywhere on Ω for some $\epsilon > 0$.

A time function on Ω induces a foliation $\bigcup_{t\in[t_0,t_1]} \Sigma_t = \Omega$ of spacelike hypersurfaces according to the following definition.

Definition A surface Σ is *spacelike* if and only if $n_x \in \mathcal{I}_x^*$ for all $x \in \Sigma$, where n_x is a covector normal to Σ at x.

The metric e induces a surface area form on Σ_t, which we shall denote $d\sigma_t$ or $d\sigma$ when the context is clear. Let $n := \nabla f/|\nabla f|$ be the future oriented unit normal covector field to Σ_t for all $t \in [t_0, t_1]$. Also, define $n^\#$ to be the vector dual to n via the metric e.

For the application to Theorem 3.2, initial data is specified on Σ_{t_0} and Theorem 3.2 provides a bound on the $L^2(\Sigma_t)$ norm of the solution and its first derivatives for any t sufficiently close to t_0. To accomplish this bound, we assume that $\partial\Omega = \Sigma_{t_0} \cup \Sigma_{t_1}$ and use the divergence theorem.

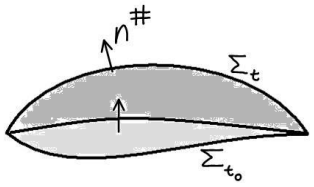


Figure 2: The geometric picture

We will now discuss the coercive properties of $(n, X) \in \mathcal{T}$ along spacelike hypersurfaces. Our goal is to show that

$$\|\partial\dot{\phi}\|^2_{L^2(\Sigma_t)} \approx \int_{\Sigma_t} \dot{Q}^\mu{}_\nu X^\nu n_\mu d\sigma.$$

From now on, we will write ${}^{(X)}J$ in place of $\dot{Q}^\mu{}_\nu X^\nu$; this is the energy current.

Definition We define the *energy current observed by* X to be

$${}^{(X)}J^\mu := \dot{Q}^\mu{}_\nu X^\nu$$

Example We consider a special case of Lemma 3.1 below, in which $h^{\mu\nu}_{AB}$ are constants and the domain is $M = \mathbb{R}^m$. Define the surface Σ_t to be the surface with the zeroth coordinate equal to t and suppose that Σ_t is spacelike. Suppose furthermore that we can pick $X \in \mathcal{J}_x$ which is constant over all $x \in M$. Provided $\xi(X) < 0$, we will make use of the positivity of the the Noether transform to show the bound

$$\|\partial\dot{\phi}\|^2_{L^2(\Sigma_t)} \lesssim \int_{\Sigma_t} {}^{(X)}J \cdot n d\sigma. \tag{16}$$

Proof Recognize that the integrand on the right hand side is simply $\frac{1}{2}\mathcal{N}(\xi, X)[\partial\dot{\phi}, \partial\dot{\phi}]$. By Plancherel's theorem, we may instead consider the Fourier transform of the integrand in the spatial coordinates. Also, the components $\mathcal{N}^{\mu\nu}_{AB}$ are constant, so we must evaluate

$$\int \mathcal{N}^{00}_{AB}\partial_t\hat{\dot{\phi}}^A\overline{\partial_t\hat{\dot{\phi}}^B} + \mathcal{N}^{m0}_{AB}i\zeta_m\hat{\dot{\phi}}^A\overline{\partial_t\hat{\dot{\phi}}^B} + \mathcal{N}^{0n}_{AB}\partial_t\hat{\dot{\phi}}^A\overline{i\zeta_n\hat{\dot{\phi}}^B} + \mathcal{N}^{mn}_{AB}i\zeta_m\hat{\dot{\phi}}^A\overline{i\zeta_n\hat{\dot{\phi}}^B}d\zeta. \tag{17}$$

where ζ is the variable corresponding to x in Fourier space and multiple lower case Latin indices indicate a summation over spatial dimensions only. The idea is to decompose $\hat{\dot{\phi}}^A = U^A + iV^A$ where U^A and V^A are real. Then $\partial_t\hat{\dot{\phi}}^A = \partial_t U^A + i\partial_t V^A$, and $\overline{\hat{\dot{\phi}}^A} = U^A - iV^A$. From the symmetries of $\mathcal{N}^{\mu\nu}_{AB}$, the imaginary components cancel in the first and fourth terms of (17). Additionally, the imaginary components associated to the middle two terms cancel each other. We are left with

$$\begin{aligned}\int \mathcal{N}^{00}_{AB}(\partial_t U^A\partial_t U^B + \partial_t V^A\partial_t V^B) + \mathcal{N}^{m0}_{AB}(\zeta_m U^A\partial_t V^B - \zeta_m V^A\partial_t U^B)\\ + \mathcal{N}^{0n}_{AB}(\partial_t V^A\zeta_n U^B - \partial_t U^A\zeta_n V^B) + \mathcal{N}^{mn}_{AB}(\zeta_m U^A\zeta_n U^B + \zeta_m V^A\zeta_n V^B)d\zeta.\end{aligned} \tag{18}$$

Setting $\zeta = \zeta_m dx^m$, we define

$$V_1 = \xi \otimes \partial_t U - \zeta \otimes V, \quad V_2 = \xi \otimes \partial_t V + \zeta \otimes U.$$

The expression (18) is equal to $\mathcal{N}[V_1, V_1] + \mathcal{N}[V_2, V_2]$ and since $V_1, V_2 \in R_\xi$, we have the following bound

$$\begin{aligned}\mathcal{N}[V_1, V_1] + \mathcal{N}[V_2, V_2] \gtrsim & |V_1|^2 + |V_2|^2\\ &= |\partial_t U|^2 + |\zeta|^2|V|^2 + |\partial_t V|^2 + |\zeta|^2|U|^2\\ &= |\partial_t\hat{\dot{\phi}}|^2 + |\zeta|^2|\hat{\dot{\phi}}|^2.\end{aligned}$$

Once again invoking Plancherel's theorem $\int_{\bar{\Sigma}_t} |\partial_t\hat{\dot{\phi}}|^2 + |\zeta|^2|\hat{\dot{\phi}}|^2 d\zeta \approx \|\partial\dot{\phi}\|^2_{L^2(\Sigma_t)}$, we arrive at the desired conclusion (16). ∎

The preceding example may be generalized to the following lemma.

Lemma 3.1 *Let Σ_t be a $m-1$ dimensional spacelike surface endowed with the surface measure $d\sigma$ which is inherited from the metric e in Ω. Let n be a unit normal covector to Σ_t and let (n, X) be a coercive pair. Then*

$$\|\partial\dot{\phi}\|^2_{L^2(\Sigma_t)} \lesssim \int_{\Sigma_t} {}^{(X)}J \cdot n d\sigma + \tilde{C} \int_{\Sigma_t} |\dot{\phi}|^2 d\sigma, \tag{19}$$

where $\tilde{C} > 0$, and the constant of $\lesssim$ depends only on X, h, $\dot{I}$, and Ω.

A very simple case in which $\tilde{C} = 0$ has already been presented in the previous example. The proof for the general case is rather involved and requires a local frequency analysis relative to a partition of unity for Σ_t. The accumulation of errors of this analysis explains the $\tilde{C}\int_{\Sigma_t}|\dot{\phi}|^2 d\sigma$ term. The full proof may be found in [Chr00].

3.2 The Main Energy Estimate

With the aid of Lemma 3.1 and the following brief observations, we shall finally be able to achieve our desired a priori estimate, Theorem 3.2.

In the introduction, it was stated that an important property of the canonical stress tensor is that its divergence can be expressed in terms of $\dot{\phi}$ and its lower order (in this case first order) derivatives. More specifically, the PDE (2) can be used to show that

$$\begin{aligned}\partial_\mu \dot{Q}^\mu{}_\nu &= \partial_\mu h^{\mu\lambda}_{AB}\partial_\lambda\dot{\phi}^B\partial_\nu\dot{\phi}^A + h^{\mu\lambda}_{AB}\partial_\mu\partial_\lambda\dot{\phi}^B\partial_\nu\dot{\phi}^A + h^{\mu\lambda}_{AB}\partial_\lambda\dot{\phi}^B\partial_\mu\partial_\nu\dot{\phi}^A \\ &\quad - \frac{1}{2}\partial_\nu h^{\alpha\beta}_{AB}\partial_\alpha\dot{\phi}^A\partial_\beta\dot{\phi}^B - \frac{1}{2}h^{\alpha\beta}_{AB}\partial_\nu\partial_\alpha\dot{\phi}^A\partial_\beta\dot{\phi}^B - \frac{1}{2}h^{\alpha\beta}_{AB}\partial_\alpha\dot{\phi}^A\partial_\nu\partial_\beta\dot{\phi}^B \qquad (20)\\ &= \dot{I}_A\partial_\nu\dot{\phi}^A + \partial_\mu h^{\mu\lambda}_{AB}\partial_\lambda\dot{\phi}^B\partial_\nu\dot{\phi}^A - \frac{1}{2}\partial_\nu h^{\alpha\beta}_{AB}\partial_\alpha\dot{\phi}^A\partial_\beta\dot{\phi}^B.\end{aligned}$$

The computation for (20) makes use of the symmetries (4) of $h^{\mu\nu}_{AB}$. Accordingly, the divergence $\partial_\mu J^\mu = \partial_\mu(\dot{Q}^\mu{}_\nu X^\nu)$ can be expressed as follows:

$$\begin{aligned}\partial_\mu J^\mu = \dot{I}_A\partial_\nu\dot{\phi}^A X^\nu + \partial_\mu h^{\mu\lambda}_{AB}\partial_\lambda\dot{\phi}^B\partial_\nu\dot{\phi}^A X^\nu - \frac{1}{2}\partial_\nu h^{\alpha\beta}_{AB}\partial_\alpha\dot{\phi}^A\partial_\beta\dot{\phi}^B X^\nu \\ + h^{\mu\lambda}_{AB}\partial_\lambda\dot{\phi}^B\partial_\nu\dot{\phi}^A\partial_\mu X^\nu - \frac{1}{2}h^{\alpha\beta}_{AB}\partial_\alpha\dot{\phi}^A\partial_\beta\dot{\phi}^B\partial_\nu X^\nu.\end{aligned} \tag{21}$$

We also recall the Cauchy-Schwartz inequality:

$$\int |\phi||\psi| \le \sqrt{\int|\phi|^2\int|\psi|^2} \le \int|\phi|^2 + \int|\psi|^2 d\mu.$$

At last, we prove the main theorem.

Theorem 3.2 *Let $\dot{\phi} = \dot{\phi}^A$ be a solution to a regularly hyperbolic PDE of the form (2) on a domain Ω endowed with a well-behaved time function f. Denote $\Sigma_t = f^{-1}(t)$ and let $d\sigma (= d\sigma_t)$ be its surface measure which is inherited from the Euclidean metric e in Ω. Let $n := \nabla f/|\nabla f|$ be the future oriented unit normal covector field to Σ_t. Then for any vectorfield X satisfying $X_x \in \mathcal{J}_x^-$, we may define a "total energy" for each level surface Σ_t given by*

$$E^2(t) := \int_{\Sigma_t} \left({}^{(X)}J^\mu n_\mu + |\dot{\phi}|^2 \right) d\sigma_t, \tag{22}$$

where ${}^{(X)}J$ is the energy current observed by X. There exists $f^{-1}([t_0, t_0 + \epsilon]) \subseteq \Omega$ and constants depending on X, h, $\dot{I}$, and ϵ such that for all $t_0 \le t \le t_0 + \epsilon$, we have the following a priori estimate.

$$E^2(t) \lesssim E^2(t_0) e^{\tilde{C}(t-t_0)} \tag{23}$$

Furthermore, by Lemma 3.1, if C is sufficiently large, then

$$\|\partial\dot{\phi}\|^2_{L^2(\Sigma_t)} + \|\dot{\phi}\|^2_{L^2(\Sigma_t)} \lesssim \left(\|\partial\dot{\phi}\|^2_{L^2(\Sigma_{t_0})} + \|\dot{\phi}\|^2_{L^2(\Sigma_{t_0})} \right) e^{\tilde{C}(t-t_0)}.$$

Remark For special cases involving symmetry, we have the much stronger a priori estimate $E(t) = E(t_0)$. This is Noether's theorem.

Proof We add a lower order term to the canonical stress. We thus define the modified current $\tilde{J} = {}^{(X)}J + J_{lo} := Q \cdot X + |\dot{\phi}|^2 n^\#$ where lo stands for "lower order". We first estimate integrals of divergence terms $\nabla \cdot {}^{(X)}J$ and $\nabla \cdot J_{lo}$ with the help of equation (21) and Lemma 3.1:

$$\begin{aligned} \left| \int_{\Sigma_t} \nabla \cdot {}^{(X)}J d\sigma \right| &\lesssim \int_{\Sigma_t} |\nabla \cdot {}^{(X)}J| d\sigma \\ &\lesssim \int_{\Sigma_t} |\partial\dot{\phi}|^2 d\sigma \\ &\lesssim \int_{\Sigma_t} {}^{(X)}J \cdot n d\sigma + C_1 \int_{\Sigma_t} |\dot{\phi}|^2 d\sigma \end{aligned} \tag{24}$$

$$\begin{aligned} \int_{\Sigma_t} \nabla \cdot J_{lo} d\sigma &= \int_{\Sigma_t} n^\# \cdot \nabla(|\dot{\phi}|^2) d\sigma + \int_{\Sigma_t} |\dot{\phi}|^2 \nabla \cdot n^\# d\sigma \\ &\lesssim \int_{\Sigma_t} |\dot{\phi}||\partial\dot{\phi}| d\sigma + \int_{\Sigma_t} |\dot{\phi}|^2 d\sigma \\ &\lesssim \int_{\Sigma_t} |\partial\dot{\phi}|^2 d\sigma + \int_{\Sigma_t} |\dot{\phi}|^2 d\sigma \\ &\lesssim \int_{\Sigma_t} {}^{(X)}J \cdot n d\sigma + (C_1 + 1) \int_{\Sigma_t} |\dot{\phi}|^2 d\sigma \end{aligned} \tag{25}$$

Combining (24) and (25), since $\int_{\Sigma_t} |\dot{\phi}|^2 d\sigma = \int_{\Sigma_t} J_{lo} \cdot n d\sigma$, we get

$$\begin{aligned}\int_{\Sigma_t} \nabla \cdot \tilde{J} d\sigma &\leq C_2 \int_{\Sigma_t} {}^{(X)}J \cdot n d\sigma + C_3 \int_{\Sigma_t} J_{lo} \cdot n d\sigma \\ &\lesssim \int_{\Sigma_t} \tilde{J} \cdot n d\sigma = E^2(t). \end{aligned} \tag{26}$$

Denote by $d\mu_e$ the volume measure on Ω associated to the metric e. From the divergence theorem and definition (22), we have

$$E^2(t_1) = E^2(t_0) + \int_{\Omega} \nabla \cdot \tilde{J} d\mu_e.$$

Then according to the co-area formula,

$$E^2(t_1) = E^2(t_0) + \int_{t_0}^{t_1} \left(\int_{\Sigma_\tau} \frac{\nabla \cdot \tilde{J}}{|\nabla f|} d\sigma_\tau \right) d\tau.$$

Assuming f is well-behaved, then

$$E^2(t_1) \leq E^2(t_0) + C \int_{t_0}^{t_1} \left(\int_{\Sigma_\tau} \nabla \cdot \tilde{J} d\sigma_\tau \right) d\tau$$

and by (26),

$$E^2(t_1) \leq E^2(t_0) + \tilde{C} \int_{t_0}^{t_1} E^2(\tau) d\tau.$$

Applying Gronwall's inequality in integral form yields the desired result (23). ∎

4 Extension to Electromagnetic Theories

The techniques in Sections 2 and 3 were presented as an application to second order PDEs. But the underlying strategy (of defining h, Q, etc. as outlined in the introduction) is not unique to second order. In this section, we will see how a similar line of argument can be applied to equations of nonlinear electromagnetism, which are first order. I will not provide proofs in this section, since many of them are similar to those in Section 2. The interested reader may consult Chapter 6 of [Chr00] for proof and [Spe10b] for construction of an actual energy estimate using the method outlined below.

The equations of nonlinear electromagnetism govern the Faraday tensor $F_{\mu\nu}$, which is a 2-form (antisymmetric tensor of type $\binom{0}{2}$), whose components relate to the electric and magnetic fields. We assume that the theory

derives from a Lagrangian $\mathcal{L}$. More precisely, the Lagrangian depends on the two invariant scalars $\mathcal{L} = \mathcal{L}(\mathcal{F}_1[F], \mathcal{F}_2[F])$ given by

$$\mathcal{F}_1[F] := \frac{1}{2} F^{\mu\nu} F_{\mu\nu},$$
$$\mathcal{F}_2[F] := \frac{1}{4} F^{\mu\nu\star} F_{\mu\nu},$$

where $\star$ denotes the Hodge dual relative to the spacetime metric. In Minkowski space under the standard foliation, the invariants $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to $|B|^2 - |E|^2$ and $E_\lambda B^\lambda$.

Although the equations[1] are expressible in many forms, the most useful for our application is

$$\begin{aligned} \nabla_{(\lambda} F_{\mu\nu)} &= {}_1 I_{\lambda\mu\nu}, \\ h^{\mu\nu\kappa\lambda} \nabla_\mu F_{\kappa\lambda} &= {}_2 I^\nu, \end{aligned} \tag{27}$$

where ${}_1 I$ and ${}_2 I$ are lower order inhomogeneous terms and the use of $()$ denotes symmetrization. As usual, the hessian is determined by

$$h^{\mu\nu\kappa\lambda} = -\frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial F_{\mu\nu} \partial F_{\kappa\lambda}}. \tag{28}$$

It is important to note the symmetries of h, namely

$$h^{\mu\nu\kappa\lambda} = -h^{\nu\mu\kappa\lambda} = -h^{\mu\nu\lambda\kappa} = h^{\kappa\lambda\mu\nu}.$$

These symmetries follow from (28), the commutativity of partial derivatives, and the fact that F is antisymmetric.

Again, we consider the linearized equations

$$\begin{aligned} \nabla_{(\lambda} \dot{F}_{\mu\nu)} &= {}_1 \dot{I}_{\lambda\mu\nu}, \\ h^{\mu\nu\kappa\lambda} \nabla_\mu \dot{F}_{\kappa\lambda} &= {}_2 \dot{I}^\nu. \end{aligned}$$

As before, these equations can be derived from a linearized Lagrangian $\dot{\mathcal{L}}$. The linearized Lagrangian leads to a canonical stress tensor

$$\dot{Q}^\mu{}_\nu := h^{\mu\zeta\kappa\lambda} \dot{F}_{\kappa\lambda} \dot{F}_{\nu\zeta} - \frac{1}{4} \delta^\mu{}_\nu h^{\zeta\eta\kappa\lambda} \dot{F}_{\zeta\eta} \dot{F}_{\kappa\lambda}.$$

For certain Lagrangian theories, such as the Maxwell-Born-Infeld theory studied in [Spe10b], this canonical stress tensor satisfies the desired properties which make it useful in constructing energy estimates: Its divergence can be expressed without derivatives of $\dot{F}$

$$\nabla_\mu \dot{Q}^\mu{}_\nu = -\frac{1}{2} h^{\zeta\eta\kappa\lambda} \dot{F}_{\zeta\eta 1} \dot{I}_{\nu\kappa\lambda} + \dot{F}_{\nu\eta 2} I^\eta + (\nabla_\mu h^{\mu\zeta\kappa\lambda}) \dot{F}_{\kappa\lambda} \dot{F}_{\nu\zeta} - \frac{1}{4} (\nabla_\nu h^{\zeta\eta\kappa\lambda}) \dot{F}_{\zeta\eta} \dot{F}_{\kappa\lambda}$$

[1] The first equation of (27) is Maxwell's equation, and ${}_1 I = 0$ corresponds to an absence of a source. The actual name of the second equation is attributed to Maxwell and the author(s) of the theory. For example, the original Maxwell equations are called the Maxwell-Maxwell equations, and the system developed by Born and Infeld is called the Maxwell-Born-Infeld system.

and $\dot{Q}^{\mu}{}_{\nu}\xi_{\mu}X^{\nu}$ is a quadratic form which can be generalized to a bilinear form acting on the space of 2-forms, $\wedge_2(T_xM)$, to which $\dot{F}$ belongs.

We may treat h as a bilinear form on $\wedge_2(T_xM)$ by $h[\theta,\omega] := h^{\mu\nu\kappa\lambda}\theta_{\mu\nu}\omega_{\kappa\lambda}$. Again, given a pair (ξ, X), we define the Noether transform to be

$$\begin{aligned}\mathcal{N}(\xi,X)[\theta,\omega] :&= h(\xi\wedge\theta\cdot X,\omega) + h(\theta,\xi\wedge\omega\cdot X) - h(\theta,\omega)\xi(X)\\ &= (h^{\xi\nu\kappa\lambda}X^{\mu} + h^{\mu\xi\kappa\lambda}X^{\nu} + h^{\mu\nu\xi\lambda}X^{\kappa} + h^{\mu\nu\kappa\xi}X^{\lambda} - h^{\mu\nu\kappa\lambda}X^{\xi})\xi_{\xi}\theta_{\mu\nu}\omega_{\kappa\lambda}\end{aligned}$$

Here, $\dot{Q}^{\mu}{}_{\nu}X^{\nu}\xi_{\mu} = \frac{1}{4}\mathcal{N}(\xi,X)[\dot{F},\dot{F}]$, and for the purpose of constructing coercive integrals, we want the Noether transform to be positive definite on

$$R_{\xi} := \{\xi\wedge\theta + \zeta\wedge\omega : \theta,\zeta,\omega\in T_x^*M\};$$

this will lead to a bound analogous to (19).

The coercive pairs (ξ, X) are those for which $\mathcal{N}(\xi, X)$ is positive definite on R_ξ and $\xi(X) < 0$. The set of these pairs has essentially the same structure as $\mathcal{T}_x$ in Section 2, and there is still a notion of characteristic subsets $\mathcal{C}_x^*$ and $\mathcal{C}_x$ which govern the sets $\mathcal{I}_x^*$ and $\mathcal{J}_x$. Introducing these characteristic subsets will be the last topic of this section.

To define $\mathcal{C}_x^* \subset T_x^*M$, we proceed as in Section 2 by introducing a matrix $\chi(\xi)$, this time acting on 1-forms $\chi^{\mu\nu}(\xi) := h^{\kappa\mu\lambda\nu}\xi_\kappa\xi_\lambda$. Due to the symmetries of h, this is really the only natural definition; up to a sign, this is the only possible matrix we could construct by contracting h with ξ twice. Previously, we defined $\mathcal{C}_x^*$ to be the set of all ξ for which $\chi(\xi)$ has a nontrivial null-space. But this definition is not useful in this application, since $\xi \in Null(\chi(\xi))$ (this fact is easy to see from the antisymmetry properties of h). Thus, we instead define $\mathcal{C}_x^*$ to be the set of ξ for which $Null(\chi(\xi)) \setminus \mathrm{span}(\xi)$ is nontrivial.

Recall that in Section 2, to define $\mathcal{C}_x$ we introduced a linear map $\psi : T_x^*M \to T_xM$ (see (9)) which depends quadratically on elements in $Null(\chi(\xi))$ for any $\xi \in \mathcal{C}_x^*$. In this application, $Null(\chi(\xi)) \subset T_x^*M$ and as mentioned in the previous paragraph, there really is only one way to construct this map. In fact, $\psi^{\mu\nu}(\xi) = \chi^{\mu\nu}(\xi)$ is the natural way, and so we define $\Lambda(\xi) := \{\chi(\theta)\cdot\xi : \theta \neq 0 \in Null(\chi(\xi))\}$. Then finally $\mathcal{C}_x := \bigcup_{\xi\in\mathcal{C}_x^*}\Lambda(\xi)$.

Of course, we have only defined the characteristic subsets $\mathcal{C}_x^*$ and $\mathcal{C}_x$, but these definitions are useful in the sense that they relate to $\mathcal{I}_x^*$ and $\mathcal{J}_x$ in the same way as in Section 2. Using the sets $\mathcal{I}_x^*$ and $\mathcal{J}_x$, we can construct energy currents which yield analogous coercive integral identities. The interested reader may find the proof in [Chr00].

Conclusion: We have seen in this paper an important relationship between the geometric and analytic properties of regularly hyperbolic PDEs. This relationship forms the basis for a variety of recent results in analysis and has potential application to a number of current problems.

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This thesis represents my own work in accordance with University regulations.



References

[AB09] L. Andersson and P. Blue. Hidden symmetries and decay for the wave equation on the Kerr spacetime. *ArXiv e-prints*, August 2009.

[Blu08] Pieter Blue. Decay of the Maxwell field on the Schwarzschild manifold. *J. Hyperbolic Differ. Equ.*, 5(4):807–856, 2008.

[BZ10] Lydia Bieri and Nina Zipser. Extensions of the stability theorem of the minkowski space in general relativity. *American Mathematical Society*, 2010.

[Chr00] Demetrios Christodoulou. *The Action Principle and Partial Differential Equations*. Princeton University Press, Princeton, NJ, 2000.

[Chr07] Demetrios Christodoulou. *The formation of shocks in 3-dimensional fluids*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.

[Chr09] Demetrios Christodoulou. *The formation of black holes in general relativity*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2009.

[CK93] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[DH06] Mihalis Dafermos and Gustav Holzegel. On the nonlinear stability of higher dimensional triaxial Bianchi-IX black holes. *Adv. Theor. Math. Phys.*, 10(4):503–523, 2006.

[DR05] Mihalis Dafermos and Igor Rodnianski. A proof of Price's law for the collapse of a self-gravitating scalar field. *Invent. Math.*, 162(2):381–457, 2005.

[DR08] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. *ArXiv e-prints*, November 2008.

[DR09] Mihalis Dafermos and Igor Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Comm. Pure Appl. Math.*, 62(7):859–919, 2009.

[Hol10a] Gustav Holzegel. On the massive wave equation on slowly rotating Kerr-AdS spacetimes. *Comm. Math. Phys.*, 294(1):169–197, 2010.

[Hol10b] Gustav Holzegel. Ultimately Schwarzschildean Spacetimes and the Black Hole Stability Problem. 2010.

[KN03] Sergiu Klainerman and Francesco Nicolò. *The evolution problem in general relativity*, volume 25 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2003.

[KR09] S. Klainerman and I. Rodnianski. On the formation of trapped surfaces. *ArXiv e-prints*, December 2009.

[LR05] Hans Lindblad and Igor Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.*, 256(1):43–110, 2005.

[LR10] Hans Lindblad and Igor Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010.

[RS09] I. Rodnianski and J. Speck. The Stability of the Irrotational Euler-Einstein System with a Positive Cosmological Constant. *ArXiv e-prints*, November 2009.

[Sid96] Thomas C. Sideris. The null condition and global existence of nonlinear elastic waves. *Invent. Math.*, 123(2):323–342, 1996.

[Spe09a] Jared Speck. The non-relativistic limit of the Euler-Nordström system with cosmological constant. *Rev. Math. Phys.*, 21(7):821–876, 2009.

[Spe09b] Jared Speck. Well-posedness for the Euler-Nordström system with cosmological constant. *J. Hyperbolic Differ. Equ.*, 6(2):313–358, 2009.

[Spe10a] J. Speck. The Global Stability of the Minkowski Spacetime Solution to the Einstein-Nonlinear Electromagnetic System in Wave Coordinates. *ArXiv e-prints*, September 2010.

[Spe10b] J. Speck. The Nonlinear Stability of the Trivial Solution to the Maxwell-Born-Infeld System. *ArXiv e-prints*, August 2010.